K-theory and t-structures

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joint work with A. Tonks (London Metropolitan) and M. Witte (Heidelberg)

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Workshop on t-structures and related topics
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R a ring

$K_0(R)$ generated by the isomorphism classes of f.g. projective R-modules mod $[P \oplus Q] = [P] + [Q]$ [Grothendieck'57]

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Quillen's K-theory

$\mathbf{E}$ an exact category

$K_0(\mathbf{E})$ generated by the objects in $\mathbf{E}$ mod $[B] = [B/A] + [A]$ for each short exact sequence $A \rightarrowtail B \twoheadrightarrow B/A$ [Grothendieck'57]

$K_n(\mathbf{E})$ [Quillen'73] with a number of theorems allowing computations

Example

For $\mathbf{E} = \mathrm{proj}(R)$ the category of f.g. projective R-modules we recover $K_n(R)$.

For $\mathbf{E}$ the category of vector bundles over a scheme X we obtain its Quillen K-theory.

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$K_0(\mathbf{T})$ generated by the objects in $\mathbf{T}$ mod $[Y] = [C_f] + [X]$ for each exact triangle $X \xrightarrow{f} Y \to C_f \to \Sigma X$

$K_n(\mathbf{T})$? several definitions by [Neeman'97–01]

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For $\mathbf{T} = D^b(\mathbf{E})$ the bounded derived category of an exact category defn *we have $K_0(\mathbf{E}) \cong K_0(C^b(\mathbf{E})) \cong K_0(D^b(\mathbf{E}))$.*

If $\mathbf{T}$ has a bounded non-degenerate t-structure with heart $\mathbf{A}$ then $K_0(\mathbf{A}) \cong K_0(\mathbf{T})$ K_0*.*

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- There's no higher K-theory satisfying agreement and localization

$$\mathbf{S} \rightarrowtail \mathbf{T} \twoheadrightarrow \mathbf{T}/\mathbf{S} \rightsquigarrow \cdots \to K_n(\mathbf{S}) \to K_n(\mathbf{T}) \to K_n(\mathbf{T}/\mathbf{S}) \to K_{n-1}(\mathbf{S}) \to \cdots$$

- There's no higher K-theory satisfying agreement for $n = 1$ and additivity

$$\left.\begin{array}{r} F, G, H\colon \mathbf{S} \longrightarrow \mathbf{T} \\ F \to G \to H \to \Sigma F \end{array}\right\} \rightsquigarrow K_n(G) = K_n(H) + K_n(F)\colon K_n(\mathbf{S}) \to K_n(\mathbf{T})$$

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Neeman's K-theories of triangulated categories

$K_n^d(\mathbf{T})$ based on the notions of exact or distinguished triangle and special octahedron ▸ defn

$K_n^v(\mathbf{T})$ based on the notions of virtual triangle ▸ defn and virtual octahedron ▸ defn

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- They are all isomorphisms for $n = 0$
- $K_n(\mathbf{A}) \cong K_n^w(\mathbf{T})$ [Neeman'98]
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$$\begin{tikzcd}
K_n(\mathbf{E}) \arrow[r] \arrow[rr, bend right, "\text{natural}"'] & K_n^{w}(D^b(\mathbf{E})) \arrow[r] & K_n^{d}(D^b(\mathbf{E})) \arrow[r, "\text{natural}"] & K_n^{v}(D^b(\mathbf{E}))
\end{tikzcd}$$

and if $\mathbf{T}$ has a t-structure with heart $\mathbf{A}$, e.g. $\mathbf{T} = D^b(\mathbf{A})$

$$\begin{tikzcd}
K_n(\mathbf{A}) \arrow[r, "\text{if defined}"] \arrow[rr, bend right, "\text{natural}"'] & K_n^{w}(\mathbf{T}) \arrow[r, "\text{if defined}"] & K_n^{d}(\mathbf{T}) \arrow[r, "\text{natural}"] & K_n^{v}(\mathbf{T})
\end{tikzcd}$$

- They are all isomorphisms for $n = 0$
- $K_n(\mathbf{A}) \cong K_n^{w}(\mathbf{T})$ [Neeman'98]
- $K_n(\mathbf{A})$ is a direct summand of $K_n^{d}(\mathbf{T})$ and $K_n^{v}(\mathbf{T})$ [Neeman'00]
- *"Very embarrasingly, this is all we know. The first question would be. . . what happens for $n = 1$?"* [Neeman'05]

Computing K_0 and K_1 simultaneously

For the different K-theories, we are going to define a chain complex of non-abelian groups $\mathcal{D}_*$ concentrated in dimensions $n = 0, 1$ whose homology is $H_n\mathcal{D}_* \cong K_n$.

$$\begin{tikzcd}
 & \mathcal{D}_0^{ab} \otimes \mathcal{D}_0^{ab} \arrow[d, "\langle\cdot,\cdot\rangle"'] & & \\
K_1 \arrow[r, tail] & \mathcal{D}_1 \arrow[r, "\partial"] & \mathcal{D}_0 \arrow[r, two heads] & K_0
\end{tikzcd}$$

The bracket $\langle\cdot,\cdot\rangle$ controls commutators in $\mathcal{D}_0$ and $\mathcal{D}_1$ as well as the action of the Hopf map

$$\begin{aligned} \eta\colon K_0 \otimes \mathbb{Z}/2 &\longrightarrow K_1 \\ x \otimes 1 &\mapsto \langle x, x\rangle \end{aligned}$$

Abelian 2-groups

An abelian 2-group C_* consists of a diagram of groups

$$C_0^{ab} \otimes C_0^{ab} \xrightarrow{\langle\cdot,\cdot\rangle} C_1 \xrightarrow{\partial} C_0$$

such that

$$\begin{aligned} \langle a,b\rangle &= -\langle b,a\rangle, & a,b &\in C_0; \\ \partial\langle a,b\rangle &= -b-a+b+a; & & \\ \langle \partial c, \partial d\rangle &= -d-c+d+c, & c,d &\in C_1. \end{aligned}$$

The homology groups of C_* are

$$\begin{aligned} H_0 C_* &= C_0/\partial(C_1), \\ H_1 C_* &= \operatorname{Ker}\partial. \end{aligned}$$

Notice that C_0 and C_1 have nilpotency class 2 and $H_0 C_*$ and $H_1 C_*$ are abelian. The group C_0 acts on C_1, $c^a = c + \langle a, \partial c\rangle$.

examples

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Abelian 2-groups

A morphism of abelian 2-groups $f_*: C_* \to D_*$ is a commutative diagram

$$\begin{tikzcd} C_0^{ab} \otimes C_0^{ab} \arrow[r, "\langle\cdot,\cdot\rangle"] \arrow[d, "f_0^{ab}\otimes f_0^{ab}"'] & C_1 \arrow[r, "\partial"] \arrow[d, "f_1"'] & C_0 \arrow[d, "f_0"'] \\ D_0^{ab} \otimes D_0^{ab} \arrow[r, "\langle\cdot,\cdot\rangle"] & D_1 \arrow[r, "\partial"] & D_0 \end{tikzcd}$$

A homotopy $\alpha: f_* \Rightarrow g_*$ is a function $\alpha: C_0 \to D_1$ satisfying

$$\alpha(a+b) = \alpha(a)^{g_0(b)} + \alpha(b),$$
$$\partial\alpha(a) = -g_0(a) + f_0(a),$$
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K_0 and K_1 of an exact category

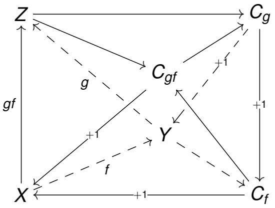
We define the abelian 2-group $\mathcal{D}_*\mathbf{E}$ by generators and relations:

n	Generators	Relations
0	$[A]$ for any object in $\mathbf{E}$	$\partial[A \rightarrowtail B \twoheadrightarrow B/A] = -[B] + [B/A] + [A]$
1	$[A \rightarrowtail B \twoheadrightarrow B/A]$ for any short exact sequence	$[B \rightarrowtail C \twoheadrightarrow C/B] + [A \rightarrowtail B \twoheadrightarrow B/A] =$ $[A \rightarrowtail C \twoheadrightarrow C/A] + [B/A \rightarrowtail C/A \twoheadrightarrow C/B]^{[A]}$ for any 2-step filtration $\begin{tikzcd} & & C/B \\ & B/A \arrow[r, tail] & C/A \arrow[u, two heads] \\ A \arrow[r, tail] & B \arrow[r, tail] \arrow[u, two heads] & C \arrow[u, two heads] \end{tikzcd}$
1		$\langle [A], [B] \rangle = -[B \rightarrowtail A \oplus B \twoheadrightarrow A]$ $+[A \rightarrowtail A \oplus B \twoheadrightarrow B]$
1		$[0 \rightarrowtail 0 \twoheadrightarrow 0] = 0$

K_0^d and K_1^d of a triangulated category

The abelian 2-group $\mathcal{D}_*^d\mathbf{T}$ is defined by generators and relations:

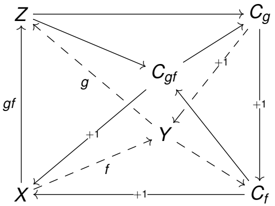
n	Generators	Relations
0	$[X]$ for any object	$\partial[X \to Y \to C_f \to \Sigma X] = -[Y] + [C_f] + [X]$
1	$[X \to Y \to C_f \to \Sigma X]$ for any exact or distinguised $\triangle$	$[Y \to Z \to C_g \to \Sigma Y] + [X \to Y \to C_f \to \Sigma X] =$ $[X \to Z \to C_{gf} \to \Sigma X] + [C_f \to C_{gf} \to C_g \to \Sigma C_f]^{[X]}$ for any special octahedron



n	Generators	Relations
1		$\langle [X], [Y] \rangle = -[Y \to X \oplus Y \to X \xrightarrow{0} \Sigma Y]$ $+[X \to X \oplus Y \to Y \xrightarrow{0} \Sigma X]$
1		$[0 \to 0 \to 0 \to \Sigma 0] = 0$

$K_0^{\vee}$ and $K_1^{\vee}$ of a triangulated category

The abelian 2-group $\mathcal{D}_*^{\vee}\mathbf{T}$ is defined by generators and relations:

n	Generators	Relations
0	$[X]$ for any object	$\partial[X \to Y \to C_f \to \Sigma X] = -[Y] + [C_f] + [X]$
1	$[X \to Y \to C_f \to \Sigma X]$ for any virtual $\triangle$	$[Y \to Z \to C_g \to \Sigma Y] + [X \to Y \to C_f \to \Sigma X] =$ $[X \to Z \to C_{gf} \to \Sigma X] + [C_f \to C_{gf} \to C_g \to \Sigma C_f]^{[X]}$ for any virtual octahedron 
1		$\langle [X],[Y]\rangle = -[Y \to X \oplus Y \to X \xrightarrow{0} \Sigma Y] + [X \to X \oplus Y \to Y \xrightarrow{0} \Sigma X]$
1		$[0 \to 0 \to 0 \to \Sigma 0] = 0$

The theorem of the heart for K_1

Theorem A

If $\mathbf{T}$ *is a triangulated category with a bounded non-degenerate* t*-structure with heart* $\mathbf{A}$*,*

$$K_1(\mathbf{A}) \xrightarrow{\cong} K_1^{d}(\mathbf{T}) \xrightarrow{\cong} K_1^{v}(\mathbf{T}).$$

Corollary

If $\mathbf{Sp}^b$ *is the stable homotopy category of spectra* X *such that* $\bigoplus_{n\in\mathbb{Z}} \pi_n X$ *is a f.g. abelian group,* $K_1^{?}(\mathbf{Sp}^b) \cong K_1^{?}(D^b(\mathbb{Z})) \cong K_1(\mathbb{Z}) = \mathbb{Z}/2.$

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The idea is to construct a strong deformation retraction

$$\begin{tikzcd} \mathcal{D}^{?}_{*}(\mathbf{T}) \arrow[loop left, "\alpha"] \arrow[r, shift left, "p_*"] & \mathcal{D}_*(\mathbf{A}), \arrow[l, shift left, "i_*"] \end{tikzcd} \qquad p_* i_* = \mathrm{id}, \qquad \alpha\colon i_* p_* \Rightarrow \mathrm{id},$$

$$p_0[X] = \cdots - [H_{-1}X] + [H_0X] - [H_1X] + \cdots .$$

An exact triangle $X \xrightarrow{f} Y \to C_f \to \Sigma X$ induces a long exact sequence

$$\cdots \to H_nX \longrightarrow H_nY \longrightarrow H_nC_f \longrightarrow H_{n-1}X \to \cdots$$

that we reindex

$$\cdots \to A_{m-1} \xrightarrow{\phi_{m-1}} A_m \xrightarrow{\phi_m} A_{m+1} \xrightarrow{\phi_{m+1}} A_{m+1} \to \cdots$$

with $A_0 = H_0Y$ and

$$p_1[X \xrightarrow{f} Y \to C_f \to \Sigma X] = \sum_{m\in\mathbb{Z}} (-1)^m[\operatorname{Ker}\phi_m \rightarrowtail A_m \twoheadrightarrow \operatorname{Ker}\phi_{m+1}]$$

$$\mod \langle \mathcal{D}_0\mathbf{A}, \mathcal{D}_0\mathbf{A}\rangle .$$

Ideas in the proof of Theorem A

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Ideas in the proof of Theorem A

The definition of p_0 is forced by the following exact triangles, $X \in \mathbf{T}_{\geq n}$,

$$X_{\geq n+1} \to X \to \Sigma^n H_n X \to \Sigma X_{\geq n+1}.$$

A truncation of an exact triangle $X \xrightarrow{f} Y \to C_f \to \Sigma X$ in $\mathbf{T}_{\geq n}$ is a special octahedron

Theorem (Vaknin'01)

There is always a truncation of an exact triangle.

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& Y' \arrow[uul, dashed] \arrow[dr, dashed] & \\
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A truncation of a virtual triangle $X \xrightarrow{f} Y \to C_f \to \Sigma X$ in $\mathbf{T}_{\geq n}$ is a virtual octahedron

$$\begin{tikzcd}
Y \arrow[rr] \arrow[dr] & & \Sigma^n H_n C_f \arrow[ddd] \arrow[ddl, dashed] \\
& C_f \arrow[ur] \arrow[ddl] & \\
& Y' \arrow[uul, dashed] \arrow[dr, dashed] & \\
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Examples in the absence of t-structures

The natural comparison homomorphism $K_1(\mathbf{E}) \to K_1^d(D^b(\mathbf{E}))$ need not always be an isomorphism.

This example goes back to Deligne, [Vaknin'01] and [Breuning'08].

Let $\mathbf{E} = \text{proj}(R)$ be the category of f.g. free modules over $R = k[\epsilon]/\epsilon^2$, k a field.

Theorem B

$K_1(\mathbf{E}) = R^\times = k \times k^\times$ but $K_1^d(D^b(\mathbf{E})) = k^\times$ and $K_1(\mathbf{E}) \to K_1^d(D^b(\mathbf{E}))$ is the projection onto the second factor.

$$\begin{tikzcd}
\mathcal{D}_*(\text{proj}(R)) \arrow[r] \arrow[d, "\text{induced by } -\otimes_R k"'] & \mathcal{D}_*^d(D^b(\text{proj}(R))) \arrow[d, "\sim"] \\
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Key ingredient in the proof of Theorem B

The element $(x, 0) \in K_1(\mathbf{E})$ is $[R \overset{1+x\epsilon}{\rightarrowtail} R \twoheadrightarrow 0] \in \mathcal{D}_1(\mathrm{proj}(R))$ and its image in $\in K_1^d(D^b(\mathbf{E}))$ is $[R \xrightarrow{1+x\epsilon} R \to 0 \to \Sigma R] \in \mathcal{D}_1^d(D^b(\mathrm{proj}(R)))$.

This element is zero by the following relations:

$$
\begin{aligned}
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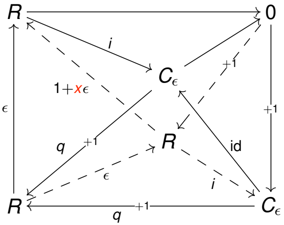
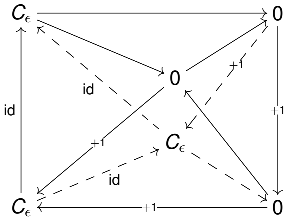
$$C_\epsilon = \quad \cdots \to 0 \to R \xrightarrow{\epsilon} R \to 0 \to \cdots$$

The element $(x,0)\in K_1(\mathbf{E})$ is $[R \overset{1+x\epsilon}{\rightarrowtail} R \twoheadrightarrow 0]\in \mathcal{D}_1(\mathrm{proj}(R))$ and its image in $\in K_1^d(D^b(\mathbf{E}))$ is $[R \xrightarrow{1+x\epsilon} R \to 0 \to \Sigma R]\in \mathcal{D}_1^d(D^b(\mathrm{proj}(R)))$.
This element is zero by the following relations:





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Examples in the absence of t-structures

The comparison homomorphism $K_1^d(\mathbf{T}) \to K_1^v(\mathbf{T})$ need not be an isomorphism.

If k is a field of char $k = 2$, $\mathbf{T} = D^b(kA_2)/\nu$ is the category f.g. free modules over $R = k[\epsilon]/\epsilon^2$, $\Sigma =$ the identity, and a 3-periodic exact sequence is an exact triangle iff it is the direct sum of a contractible triangle and a triangle of the following form

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Theorem C

$K_0^d(\mathbf{T}) = K_0^v(\mathbf{T}) = 0 = K_1^d(\mathbf{T})$ *but there is a surjective homomorphism* $\det\colon K_1^v(\mathbf{T}) \twoheadrightarrow k^\times/(k^\times)^2$.

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Sketch of the proof of Theorem C

The abelian 2-group $\mathcal{D}_*^d(\mathbf{T})$ admits a contraction α defined by

$$\alpha[P] = [P \xrightarrow{\epsilon} P \xrightarrow{\epsilon} P \xrightarrow{\epsilon} P].$$

We are going to define a morphism

$$\det\colon \mathcal{D}_*^v(\mathbf{T}) \longrightarrow (0 \xrightarrow{\langle\cdot,\cdot\rangle} k^\times/(k^\times)^2 \to 0)$$

which induces the claimed surjection.

Lemma

Virtual triangles in $\mathbf{T}$ *are 3-periodic exact sequences*

$$V = \begin{tikzcd} P_0 \arrow[rr, "d_2"] & & P_2 \arrow[dl, "d_1"] \\ & P_1 \arrow[ul, "d_0"] & \end{tikzcd}$$

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$$\epsilon V \rightarrowtail V \twoheadrightarrow \epsilon V$$

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$$\delta_n \colon H_{n+1}(\epsilon V) \cong H_n(\epsilon V).$$

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& \epsilon P_1 \arrow[ul, "\epsilon d_0"] \arrow[dd, tail] & \\
P_0 \arrow[rr, dashed, "d_2"] \arrow[dd, two heads, "\epsilon"'] & & P_2 \arrow[dl, "d_1"] \arrow[dd, two heads, "\epsilon"] \\
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$$\det(V) = \det(\delta_n \delta_{n+1} \delta_{n+2} \colon H_n(\epsilon V) \cong H_n(\epsilon V)).$$

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A virtual triangle V is exact iff $\det(V) = 1$.

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Given $u \in k^\times$, $\det(R \xrightarrow{\epsilon} R \xrightarrow{\epsilon} R \xrightarrow{u\epsilon} R) = u$.

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Given a virtual octahedron as below we have

$$\det(Y \xrightarrow{g} Z \to C_g \to Y)\det(X \xrightarrow{f} Y \to C_f \to X)$$
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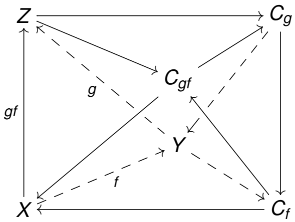

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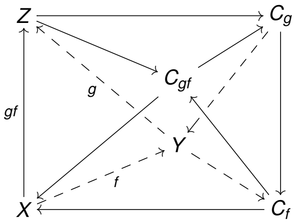

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K-theory and t-structures

Fernando Muro

Universidad de Sevilla

joint work with A. Tonks (London Metropolitan) and M. Witte (Heidelberg)

`arXiv:1006.5399v1 [math.KT]`

Workshop on t-structures and related topics
Stuttgart, July 7–9, 2011

Example

The inverse isomorphism of

$$\begin{aligned} K_0(\mathbf{E}) &\longrightarrow K_0(C^b(\mathbf{E})) \\ [A] &\mapsto [\cdots \to 0 \to A \to 0 \to \cdots] \end{aligned}$$

is the Euler characteristic,

$$\begin{aligned} K_0(C^b(\mathbf{E})) &\longrightarrow K_0(\mathbf{E}) \\ [\cdots \to X_n \to X_{n-1} \to \cdots] &\mapsto \sum_{n\in\mathbb{Z}} (-1)^n [X_n] \end{aligned}$$

back

The bounded derived category of an exact category

Let **E** be an idempotent complete exact category and **A** its abelian envelope. The bounded homotopy category $K^b(\mathbf{E})$ contains a thick subcategory $A^b(\mathbf{E})$ spanned by the acyclic bounded complexes

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such that $\operatorname{Ker} d_n \in \mathbf{E}$ for all $n \in \mathbb{Z}$.

The bounded derived category of **E** [Neeman'90] is defined as the Verdier quotient

$$D^b(\mathbf{E}) = K^b(\mathbf{E})/A^b(\mathbf{E}).$$

Example

$D^b(\mathbf{A})$ is the usual derived category of an abelian category.

If R is a ring, $D^b(\mathrm{proj}(R)) = K^b(\mathrm{proj}(R))$.

$D(\mathrm{Flat}(R)) \simeq K(\mathrm{Proj}(R))$ [Neeman'08] back.

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The inverse isomorphism of

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is given by

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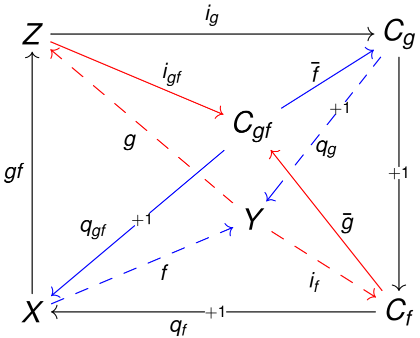
since $\mathbf{T} = \bigcup_{n\in\mathbb{Z}} \mathbf{T}_{\geq n}$ *and if* $X \in \mathbf{T}_{\geq n}$ *the exact triangle*

$$X_{\geq n+1} \to X \to \Sigma^n H_nX \to \Sigma X_{\geq n+1}$$

yields the relation $[X] = (-1)^n[H_nX] + [X_{\geq n+1}]$. back

Special octahedra

An octahedron made up from exact triangles



is special if the following triangles are exact back

$$Y \xrightarrow{\binom{g}{-i_f}} Z \oplus C_f \xrightarrow{(i_{gf},\, \bar g)} C_{gf} \xrightarrow{q_g \bar f} \Sigma Y$$

$$C_{gf} \xrightarrow{\binom{q_{gf}}{-\bar f}} \Sigma X \oplus C_g \xrightarrow{(\Sigma f,\, q_g)} \Sigma Y \xrightarrow{\Sigma(\bar g q_f)} \Sigma C_{gf}$$

Virtual triangles

A triangle is contractible if it is a direct sum of triangles of the form

$$X \xrightarrow{1} X \to 0 \to \Sigma X, \quad 0 \to Y \xrightarrow{1} Y \to 0, \quad Z \to 0 \to \Sigma Z \xrightarrow{1} \Sigma Z.$$

A triangle is virtual if it is a direct summand with contractible complement of a triangle

$$X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{q} \Sigma X$$

such that we can replace each arrow by another morphism to obtain an exact triangle.

Example

If $X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{q} \Sigma X$ is an exact triangle then $X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{-q} \Sigma X$ is a virtual triangle, but in general not exact.

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Virtual triangles

A triangle is contractible if it is a direct sum of triangles of the form

$$X \xrightarrow{1} X \to 0 \to \Sigma X, \qquad 0 \to Y \xrightarrow{1} Y \to 0, \qquad Z \to 0 \to \Sigma Z \xrightarrow{1} \Sigma Z.$$

A triangle is virtual if it is a direct summand with contractible complement of a triangle

$$X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{q} \Sigma X$$

such that we can replace each arrow by another morphism to obtain an exact triangle.

Example

If $X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{q} \Sigma X$ is an exact triangle then $X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{-q} \Sigma X$ is a virtual triangle, but in general not exact.

back

Virtual octahedra

An octahedron made up from virtual triangles

$$
\begin{tikzcd}
Z \arrow[rr, "i_g"] \arrow[dr, "i_{gf}"] & & C_g \arrow[ddl, dashed, "q_g", "+1"'] \arrow[ddd, "+1"] \\
& C_{gf} \arrow[ur, "\bar f"] \arrow[ddl, "q_{gf}"', "+1"] & \\
& Y \arrow[uul, dashed, "g"'] \arrow[dr, dashed, "i_f"'] & \\
X \arrow[uuu, "gf"] \arrow[ur, dashed, "f"'] & & C_f \arrow[uul, "\bar g"'] \arrow[ll, "q_f", "+1"']
\end{tikzcd}
$$

is virtual if the following triangles are virtual back

$$
Y \xrightarrow{\binom{g}{-i_f}} Z \oplus C_f \xrightarrow{(i_{gf},\, \bar g)} C_{gf} \xrightarrow{q_g \bar f} \Sigma Y
$$

$$
C_{gf} \xrightarrow{\binom{q_{gf}}{-\bar f}} \Sigma X \oplus C_g \xrightarrow{(\Sigma f,\, q_g)} \Sigma Y \xrightarrow{\Sigma(\bar g q_f)} \Sigma C_{gf}
$$

Examples of abelian 2-groups

Proposition

The 2-category of abelian 2-groups is 2-equivalent to the 2-category of Picard groupoids.

- Abelian group homomorphism $B \otimes B \xrightarrow{0} A \xrightarrow{f} B$ [Deligne'63–64]
- $0 \xrightarrow{\langle\cdot,\cdot\rangle} \mathbb{Z} \longrightarrow 0$ is freely generated by $1 \in \mathbb{Z}$ in degree $n = 1$
- $\mathbb{Z} \otimes \mathbb{Z} \overset{\langle\cdot,\cdot\rangle}{\twoheadrightarrow} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}$ is freely generated by $1 \in \mathbb{Z}$ in degree $n = 0$, it is quasi-isomorphic to $\mathcal{D}_*(\text{proj}(\mathbb{Z}))$
- $\mathbb{Z} \otimes \mathbb{Z} \overset{\langle\cdot,\cdot\rangle}{\twoheadrightarrow} R^\times \xrightarrow{0} \mathbb{Z}$, R a commutative local ring, $\langle 1, 1\rangle = 1 \in R^\times$, it is quasi-isomorphic to $\mathcal{D}_*(\text{proj}(R))$
- If G is a group of nilpotency class two and $H \subset G$ is a subgroup containing the commutators $[G, G] \subset H$ back

$$G^{ab} \otimes G^{ab} \xrightarrow{\langle\cdot,\cdot\rangle} H \xrightarrow{\text{incl.}} G$$
$$\langle a, b\rangle = -b - a + b + a$$

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