Picard categories, determinant functors and K-theory

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Additive invariants

An additive invariant of an exact category $\mathcal{E}$ with values in an abelian group G is a function $\varphi\colon \operatorname{Ob}\mathcal{E} \to G$ taking short exact sequences $A \rightarrowtail B \twoheadrightarrow B/A$ to sums,

$$\varphi(B) = \varphi(B/A) + \varphi(A).$$

The universal additive invariant is

$$\operatorname{Ob}\mathcal{E} \to K_0(\mathcal{E})\colon A \mapsto [A],$$

i.e. let $\operatorname{Add}(\mathcal{E}, G)$ be the set of additive invariants taking values in G. The functor

$$\operatorname{Add}(\mathcal{E}, -)\colon \mathbf{Ab} \to \mathbf{Set}$$

is represented by $K_0(\mathcal{E})$.

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Example

Let $\mathcal{E} = \mathbf{vect}(X)$ be the category of vector bundles over a space or scheme X.

$$\begin{aligned} E &\mapsto \operatorname{rank} E \in H^0(X, \mathbb{Z}), \\ E &\mapsto \wedge^{\operatorname{rank} E} E \in \operatorname{Pic}(X), \ \textit{the determinant line bundle}. \end{aligned}$$

Determinant functors

A Picard groupoid is a symmetric monoidal groupoid $\mathcal{G}$ such that

$$V \otimes - \colon \mathcal{G} \xrightarrow{\sim} \mathcal{G}$$

is an equivalence for any object V. A determinant functor is a functor

$$\varphi \colon \mathcal{E}^{\text{iso}} \longrightarrow \mathcal{G}$$

satisfying:

Additivity. For each short exact sequence $A \rightarrowtail B \twoheadrightarrow B/A$ there is a morphism

$$\varphi(A \rightarrowtail B \twoheadrightarrow B/A) \colon \varphi(B) \longrightarrow \varphi(B/A) \otimes \varphi(A),$$

natural with respect to isomorphisms of short exact sequences.

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Determinant functors

Associativity. For each 2-step filtration $A \rightarrowtail B \rightarrowtail C$

$$\begin{tikzcd}
 & \varphi(C) \arrow[dl, "\varphi(B \rightarrowtail C \twoheadrightarrow C/B)"'] \arrow[dr, "\varphi(A \rightarrowtail C \twoheadrightarrow C/A)"] & \\
\varphi(C/B) \otimes \varphi(B) \arrow[d, "1 \otimes \varphi(A \rightarrowtail B \twoheadrightarrow B/A)"'] & & \varphi(C/A) \otimes \varphi(A) \arrow[d, "\varphi(B/A \rightarrowtail C/A \twoheadrightarrow C/B) \otimes 1"'] \\
\varphi(C/B) \otimes (\varphi(B/A) \otimes \varphi(A)) \arrow[rr, "\text{asoc. of } \otimes"'] & & (\varphi(C/B) \otimes \varphi(B/A)) \otimes \varphi(A)
\end{tikzcd}$$

Commutativity.

$$\begin{tikzcd}
 & \varphi(A \oplus B) \arrow[dl, "\varphi(B\rightarrowtail A\oplus B\twoheadrightarrow A)"'] \arrow[dr, "\varphi(A\rightarrowtail A\oplus B\twoheadrightarrow B)"] & \\
\varphi(A) \otimes \varphi(B) \arrow[rr, "\text{comm. of } \otimes"'] & & \varphi(B) \otimes \varphi(A)
\end{tikzcd}$$

Determinant functors

Compatibility with 0, there is a counit morphism $\varphi_0\colon \varphi(0) \to I$, where I is the unit object of $\mathcal{G}$, such that the following composites are identity morphisms

$$\varphi(A) \xrightarrow{\varphi(0 \rightarrowtail A \twoheadrightarrow A)} \varphi(A) \otimes \varphi(0) \xrightarrow{1 \otimes \varphi_0} \varphi(A) \otimes I \xrightarrow{\text{unit of } \otimes} \varphi(A)\ ,$$

$$\varphi(A) \xrightarrow{\varphi(A \rightarrowtail A \twoheadrightarrow 0)} \varphi(0) \otimes \varphi(A) \xrightarrow{\varphi_0 \otimes 1} I \otimes \varphi(A) \xrightarrow{\text{unit of } \otimes} \varphi(A)\ .$$

Example

Let $\mathbf{Pic}(X)$ *be the Picard groupoid of graded line bundles* (L, n)*, given by a line bundle* L *over* X *and a locally constant function* $n\colon X \to \mathbb{Z}$.

$$(L, n) \otimes (M, p) \quad = \quad (L \otimes M, n + p).$$

There is a determinant functor

$$\det\colon \mathbf{vect}(X)^{\text{iso}} \longrightarrow \mathbf{Pic}(X)$$

defined by the graded determinant line bundle

$$\det E \quad = \quad \left(\wedge^{\operatorname{rank} E} E, \operatorname{rank} E\right).$$

Deligne's universal determinant functor

Deligne's Picard groupoid of virtual objects $V(\mathcal{E})$ is the recipient of the universal determinant functor

$$\mathcal{E}^{\text{iso}} \longrightarrow V(\mathcal{E}).$$

Let **PG** be the 2-category of Picard groupoids, colax symmetric monoidal functors and natural transformations. The homotopy category $\mathbf{PG}_{\simeq}$ is obtained by dividing out 2-morphisms.

Two determinant functors $\varphi, \psi \colon \mathcal{E}^{\text{iso}} \to \mathcal{G}$ are homotopic if there is a natural transformation $\varphi \Rightarrow \psi$ compatible with the additivity morphisms and the counit.

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Let $\det(\mathcal{E}, \mathcal{G})$ be the set of homotopy classes of determinant functors.

Theorem (Deligne'87)

The functor

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is represented by $V(\mathcal{E})$.

Virtual objects and K-theory

The homotopy groups of a Picard groupoid $\mathcal{G}$ are

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\begin{aligned}
\pi_0\mathcal{G} &= \text{isomorphism classes of objects, the sum is induced by } \otimes,\\
\pi_1\mathcal{G} &= \mathrm{Aut}_{\mathcal{G}}(I).
\end{aligned}
$$

Example

$$
\begin{aligned}
\pi_0\mathbf{Pic}(X) &= \mathrm{Pic}(X)\oplus H^0(X,\mathbb{Z}), & \pi_0 V(\mathcal{E}) &= K_0(\mathcal{E}),\\
\pi_1\mathbf{Pic}(X) &= \mathcal{O}_X^*(X), & \pi_1 V(\mathcal{E}) &= K_1(\mathcal{E}).
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In particular a determinant functor $\varphi\colon \mathcal{E}^{\mathrm{iso}} \to \mathcal{G}$ determines a morphism $V(\mathcal{E}) \to \mathcal{G}$ in $\mathbf{PG}_{\simeq}$ which induces homomorphisms

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K_0(\mathcal{E}) &\longrightarrow \pi_0\mathcal{G},\\
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Virtual objects and K-theory

Let $\mathbf{Spec}_{0,1}$ be the stable homotopy category of spectra with homotopy concentrated in degrees 0 and 1. There is an equivalence of categories

$$B\colon \mathbf{PG}_{\simeq} \xrightarrow{\sim} \mathbf{Spec}_{0,1}.$$

Theorem (Deligne'87)

$BV(\mathcal{E})$ is naturally isomorphic to the 1-truncation of Quillen's K-theory spectrum $K(\mathcal{E})$ in the stable homotopy category.

As a consequence

$$\det(\mathcal{E},\mathcal{G}) \cong \mathrm{Hom}_{\mathbf{PG}_{\simeq}}(V(\mathcal{E}),\mathcal{G}) \cong [K(\mathcal{E}), B\mathcal{G}].$$

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Knudsen–Mumford'76 tackled the problem of defining a functorial graded determinant line bundle for a bounded complex E^* in $\mathbf{vect}(X)$,

$$\cdots \to E^{n-1} \xrightarrow{d} E^n \xrightarrow{d} E^{n+1} \to \cdots,$$

$$\det E^* = \left(\bigotimes_{n\in\mathbb{Z}} \left(\wedge^{\operatorname{rank} E^n} E^n\right)^{(-1)^n}, \sum_{n\in\mathbb{Z}} (-1)^n \operatorname{rank} E^n\right).$$

The main difficulties were the definition of det on morphisms, as well as the additivity morphisms associated to short exact sequences, and to prove the uniqueness of det up to natural isomorphism.

Determinant functors for Waldhausen categories

One can define determinant functors on a Waldhausen category $\mathcal{W}$, such as $\mathcal{W} = C^b(\mathcal{E})$, replacing isomorphisms by weak equivalences and short exact sequences by cofiber sequences,

$$\mathcal{W}^{\text{we}} \longrightarrow \mathcal{G}.$$

Let $\det(\mathcal{W}, \mathcal{G})$ be the set of homotopy classes of determinant functors.

Theorem (M.–Tonks'07)

The functor

$$\det(\mathcal{W}, -) \colon \mathbf{PG}_{\simeq} \longrightarrow \mathbf{Set}$$

is represented by a Picard groupoid $V(\mathcal{W})$ such that $BV(\mathcal{W})$ is naturally isomorphic to the 1-truncation of Waldhausen's K-theory spectrum $K(\mathcal{W})$.

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Corollary (Knudsen'02)

The inclusion $\mathcal{E} \subset C^b(\mathcal{E})$ induces a natural isomorphism between the functors

$$\det(C^b(\mathcal{E}), -) \cong \det(\mathcal{E}, -)\colon \mathbf{PG}_{\simeq} \longrightarrow \mathbf{Set}.$$

Proof.

$$\begin{tikzcd}
\det(\mathcal{E},\mathcal{G}) \arrow[r, "\cong"] & {[K(\mathcal{E}), B\mathcal{G}]} \\
\det(C^b(\mathcal{E}),\mathcal{G}) \arrow[u] \arrow[r, "\cong"] & {[K(C^b(\mathcal{E})), B\mathcal{G}]} \arrow[u]
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The right vertical arrow is an isomorphism by the Gillet–Waldhausen theorem.

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$\square$

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We define an inverse equivalence $\mathcal{P}$

$$\mathbf{PG}_{\simeq} \underset{\mathcal{P}}{\overset{B}{\rightleftarrows}} \mathbf{Spec}_{0,1},$$

and look at the Picard groupoid $\mathcal{P}K(\mathcal{W})$. For this we use the machinery of crossed complexes.

A crossed module C_* is a group homomorphism

$$\partial\colon C_2 \longrightarrow C_1$$

together with an exponential action of C_1 on the right of C_2 such that

$$\begin{aligned}\partial(c_2^{c_1}) &= -c_1 + \partial(c_2) + c_1,\\ c_2^{\partial(d_2)} &= -d_2 + c_2 + d_2.\end{aligned}$$

Example

The crossed module of a pair of connected spaces $Y \subset X$ is

$$\partial\colon \pi_2(X, Y) \longrightarrow \pi_1(Y).$$

The fundamental crossed module of a reduced simplicial set Z is the crossed module of $|Z|^1 \subset |Z|$.

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A crossed complex C_* is a chain complex of groups

$$\cdots \to C_{n+1} \xrightarrow{\partial} C_n \to \cdots \to C_2 \xrightarrow{\partial} C_1 \to 0$$

where $\partial\colon C_2 \to C_1$ is a crossed module, C_n is an $H_1 C_*$-module for $n > 2$, and ∂ is always C_1-equivariant.

Example

The fundamental crossed complex $\pi(Z)$ of a reduced simplicial set Z is

$$\cdots \to \pi_{n+1}(|Z|^{n+1}, |Z|^n) \xrightarrow{\partial} \pi_n(|Z|^n, |Z|^{n-1}) \to \cdots$$
$$\cdots \to \pi_2(|Z|^2, |Z|^1) \xrightarrow{\partial} \pi_1(|Z|^1) \to 0.$$

Here $|Z|^n$ are the skeleta of the geometric realization $|Z|$. Notice that the fundamental crossed module is the truncation $t_{\leq 2}\pi(Z)$.

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Brown–Higgins'87 defined the tensor product of crossed complexes, which yields a closed symmetric monoidal structure.

Theorem (Tonks'93)

There is an Eilenberg-Zilber strong deformation retraction for the fundamental crossed complex of a product of simplicial sets,

$$\pi(Y) \otimes \pi(Z) \rightleftarrows \pi(Y \times Z) \circlearrowleft .$$

In particular the fundamental crossed complex of a simplicial monoid is a crossed chain algebra.

A commutative monoid in crossed modules C_* is a diagram

$$C_1 \times C_1 \xrightarrow{\langle\cdot,\cdot\rangle} C_2 \xrightarrow{\partial} C_1$$

where ∂ is a crossed module C_* and $\langle\cdot,\cdot\rangle$ lifts the commutator bracket

$$\partial\langle c_1, d_1\rangle = [d_1, c_1] = -d_1 - c_1 + d_1 + c_1 .$$

The loop Picard groupoid ΩC_* is

$$\begin{tikzcd} C_1 \ltimes C_2 \arrow[r, shift left=2, "s"] \arrow[r, "t"] & C_1 \arrow[l, shift left=2, "i"] \end{tikzcd}, \qquad \begin{array}{rcl} s(c_1, c_2) & = & c_1, \\ t(c_1, c_2) & = & c_1 + \partial(c_2), \\ i(c_1) & = & (c_1, 0). \end{array}$$

The tensor product is defined by the group structure $\otimes = +$, and the commutativity constraint is

$$(c_1 + d_1, \langle c_1, d_1\rangle) \colon c_1 + d_1 \longrightarrow d_1 + c_1 .$$

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A commutative monoid in crossed modules satisfying the law

$$\langle\cdot,[\cdot,\cdot]\rangle \quad = \quad 0$$

is a stable quadratic module, i.e. a diagram

$$C_1^{ab} \otimes C_1^{ab} \xrightarrow{\langle\cdot,\cdot\rangle} C_2 \xrightarrow{\partial} C_1$$

with

$$\begin{aligned} \partial\langle c_1, d_1\rangle &= [d_1, c_1], \\ \langle\partial(c_2), \partial(d_2)\rangle &= [d_2, c_2], \\ \langle c_1, d_1\rangle &= -\langle d_1, c_1\rangle. \end{aligned}$$

The action of C_1 on C_2 is given by $c_2^{c_1} = c_2 + \langle c_1, \partial(c_2)\rangle$. The subcategory of stable quadratic modules is reflective. ▸ example

Let $\mathbf{SQM}_f$ be the homotopy category of stable quadratic modules C_* with C_1 free in the variety of groups of nilpotency class 2. Then

$$\Omega\colon \mathbf{SQM}_f \xrightarrow{\sim} \mathbf{PG}_{\simeq}$$

is an equivalence with $\pi_i \Omega C_* = H_{i+1} C_*$, $i = 0, 1$. We now define the equivalence $\mathcal{P}$ as a composite

$$\begin{tikzcd}
\mathbf{Spec}_{0,1} \arrow[r, "\mathcal{P}", "\sim"'] \arrow[d, "\text{incl.}"'] & \mathbf{PG}_{\simeq} \\
\mathbf{Spec}_{\geq 0} \arrow[r, "\lambda"] & \mathbf{SQM}_f \arrow[u, "\sim", "\Omega"']
\end{tikzcd}$$

Let us define

$$\lambda\colon \mathbf{Spec}_{\geq 0} \longrightarrow \mathbf{SQM}_f.$$

A connective spectrum X is a sequence of pointed simplicial sets and connecting maps,

$$X_0, X_1, \dots, X_n, \dots; \quad \Sigma X_n \longrightarrow X_{n+1}.$$

$$\begin{tikzcd}
\begin{array}{c} X_1 \text{ reduced} \\ \text{unital} \\ E_\infty\text{-monoid} \end{array} \arrow[r, dashed, "\text{take } \pi", "\text{E-Z}"'] & \begin{array}{c} \pi(X_1) \text{ crossed} \\ E_\infty\text{-algebra} \end{array} \arrow[r, dashed, "\text{truncate}", "t_{\leq 2}"'] & \begin{array}{c} t_{\leq 2}\pi(X_1) \text{ comm.} \\ \text{monoid} \end{array} \arrow[d, dashed, "\text{reflect}"] \\
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The fundamental Picard groupoid of a spectrum

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$$\begin{array}{ccccc} \text{X_1 reduced unital E_∞-monoid} & \xrightarrow[\text{E-Z}]{\text{take }\pi} & \pi(X_1)\text{ crossed E_∞-algebra} & \xrightarrow[t_{\leq 2}]{\text{truncate}} & t_{\leq 2}\pi(X_1)\text{ comm. monoid} \\ &&&& \downarrow \text{reflect} \\ &&&& \lambda(X)\text{ stable quadratic module} \end{array}$$

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The fundamental Picard groupoid of a spectrum

Let us define

$$\lambda\colon \mathbf{Spec}_{\geq 0} \longrightarrow \mathbf{SQM}_f.$$

A connective spectrum X is a sequence of pointed simplicial sets and connecting maps,

$$X_0, X_1, \dots, X_n, \dots; \quad \Sigma X_n \longrightarrow X_{n+1}.$$

$$\begin{tikzcd}[column sep=large]
\begin{array}{c} X_1 \text{ reduced} \\ \text{unital} \\ E_\infty\text{-monoid} \end{array} \arrow[r, dashed, "\text{take } \pi", "\text{E-Z}"'] & \begin{array}{c} \pi(X_1) \text{ crossed} \\ E_\infty\text{-algebra} \end{array} \arrow[r, dashed, "\text{truncate}", "t_{\leq 2}"'] & \begin{array}{c} t_{\leq 2}\pi(X_1) \text{ comm.} \\ \text{monoid} \end{array} \arrow[d, dashed, "\text{reflect}"] \\
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$\downarrow$ reflect

$\lambda(X)$ stable quadratic module

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Virtual objects for Waldhausen categories

It would be tempting to define $V(\mathcal{W})$ as

$$\mathcal{P}K(\mathcal{W}) \;=\; \Omega\,(\text{reflection of } t_{\leq 2}\pi(\text{diag } Y))\,, \quad Y \;=\; \text{ner}(S.\mathcal{W})^{\text{we}},$$

unfortunately this does not look like the universal recipient of determinant functors on $\mathcal{W}$.

Definition

The total crossed complex $\pi^{\text{tot}}(Y)$ of a bisimplicial set Y is

$$\cdots \to \pi_{n+1}(\|Y\|^{n+1}, \|Y\|^{n}) \xrightarrow{\partial} \pi_n(\|Y\|^{n}, \|Y\|^{n-1}) \to \cdots$$
$$\cdots \to \pi_2(\|Y\|^{2}, \|Y\|^{1}) \xrightarrow{\partial} \pi_1(\|Y\|^{1}) \to 0.$$

Here $\|\cdot\|$ is Segal's geometric realization.

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Here $\|\cdot\|$ *is Segal's geometric realization.*

Theorem (M.–Tonks'07)

There is an Eilenberg-Zilber-Cartier strong deformation retraction from the fundamental crossed complex of the diagonal of a bisimplicial set Y to its total crossed complex,

$$\pi^{\text{tot}}(Y) \rightleftarrows \pi(\text{diag } Y) \circlearrowleft .$$

As a consequence if we define

$$\begin{array}{rcl} V(\mathcal{W}) & = & \Omega\mathcal{D}_*(\mathcal{W}), \\ \mathcal{D}_*(\mathcal{W}) & = & \text{reflection of } t_{\leq 2}\pi^{\text{tot}}(\text{ner}(S.\mathcal{W})^{\text{we}}), \end{array}$$

then

$$V(\mathcal{W}) \rightleftarrows \mathcal{P}K(\mathcal{W}) \circlearrowleft .$$

Virtual objects for Waldhausen categories

The stable quadratic module $\mathcal{D}_*(\mathcal{W})$ is defined by generators and relations. Generators correspond to bisimplices of total degree 1 and 2 in $\mathrm{ner}(S.\mathcal{W})^{\mathrm{we}}$, relators correspond to degenerate bisimplices and bisimplices of total degree 3. There is also a relation for the bracket $\langle \cdot, \cdot \rangle$ determined by the Eilenberg-Zilber strong deformation retraction for π^{tot}.

An inspection of this presentation reveals why there is a determinant functor

$$\psi\colon \mathcal{W}^{\mathrm{we}} \longrightarrow \Omega\mathcal{D}_*(\mathcal{W}) \; = \; V(\mathcal{W})$$

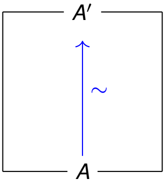
and why it is universal.

Generators of $\mathcal{D}_*(\mathcal{W})$

Bisimplices of total degree 1 and 2 in $\mathrm{ner}(S.\mathcal{W})^{\mathrm{we}}$

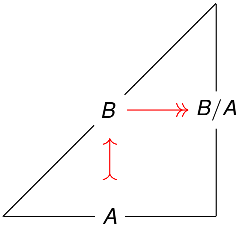
$[A]$

ψ of an object



$[A \xrightarrow{\sim} A']$

ψ of a morphism



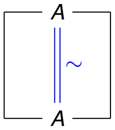
$[A \rightarrowtail B \twoheadrightarrow B/A]$

additivity

$$0$$

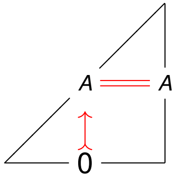
$[0] = 0$

compat. with 0



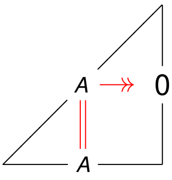
$[1_A\colon A \xrightarrow{\sim} A] = 0$

ψ preserves identities



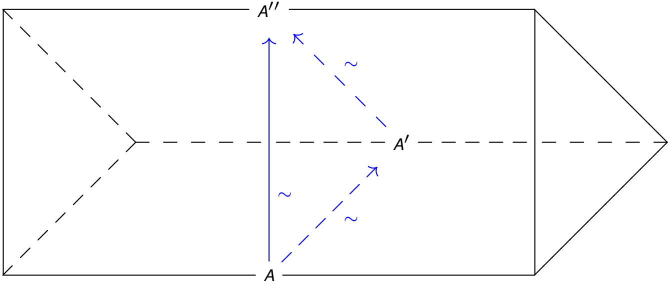
$[0 \rightarrowtail A \twoheadrightarrow A] = 0$

compat. with 0



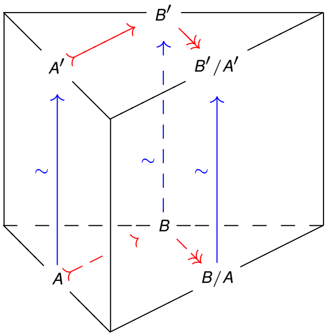
$[A \rightarrowtail A \twoheadrightarrow 0] = 0$

compat. with 0



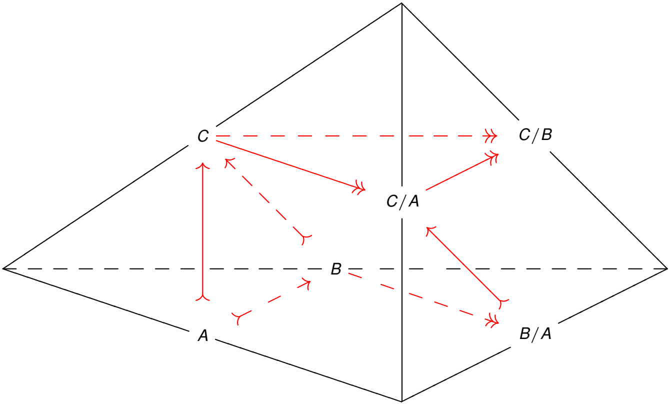
$$[A \xrightarrow{\sim} A''] = [A' \xrightarrow{\sim} A''] + [A \xrightarrow{\sim} A']$$

ψ preserves composition



$$[A' \rightarrowtail B' \twoheadrightarrow B'/A'] + [A \xrightarrow{\sim} A'] + [B/A \xrightarrow{\sim} B'/A']^{[A]} = [B \xrightarrow{\sim} B'] + [A \rightarrowtail B \twoheadrightarrow B/A]$$

additivity morphisms are natural

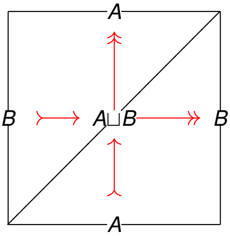


$$[B \rightarrowtail C \twoheadrightarrow C/B] + [A \rightarrowtail B \twoheadrightarrow B/A] = [A \rightarrowtail C \twoheadrightarrow C/A] + [B/A \rightarrowtail C/A \twoheadrightarrow C/B]^{[A]}$$

associativity

Relators of $\mathcal{D}_*(\mathcal{W})$

The bracket $\langle \cdot, \cdot \rangle$



$$\langle [A], [B] \rangle = -[B \rightarrowtail A \sqcup B \twoheadrightarrow A] + [A \rightarrowtail A \sqcup B \twoheadrightarrow B]$$

commutativity

Derived determinant functors

Grothendieck asked Knudsen in 1973 whether determinant functors

$$C^b(\mathcal{E})^{\mathrm{we}} \longrightarrow \mathcal{G}$$

coincide with derived determinant functors, i.e. a functor

$$\varphi\colon D^b(\mathcal{E})^{\mathrm{iso}} \longrightarrow \mathcal{G}$$

together with additivity morphisms

$$\varphi(A \rightarrowtail B \twoheadrightarrow B/A)\colon \varphi(B) \longrightarrow \varphi(B/A) \otimes \varphi(A),$$

natural with respect to isomorphisms in the derived category of extensions $D^b(\mathrm{Ext}(\mathcal{E}))^{\mathrm{iso}}$, satisfying associativity for 2-step filtrations $A \rightarrowtail B \rightarrowtail C$, as well as commutativity, and compatibility with 0.

Let $\det^{\mathrm{der}}(\mathcal{E},\mathcal{G})$ be the set of homotopy classes of derived determinant functors.

Theorem (M.'07)

The functor

$$\det^{\mathrm{der}}(\mathcal{E},-)\colon \mathbf{PG}_{\simeq} \longrightarrow \mathbf{Set}$$

is represented by a Picard groupoid $V^{\mathrm{der}}(\mathcal{E})$ such that $BV^{\mathrm{der}}(\mathcal{E})$ is naturally isomorphic to the 1-truncation of Maltsiniotis's K-theory spectrum $K(\mathbb{D}(\mathcal{E}))$ of the triangulated derivator $\mathbb{D}(\mathcal{E})$.

Derived determinant functors

As in the case of derived functors on Waldhausen categories

$$\begin{aligned} V^{\text{der}}(\mathcal{E}) &= \Omega\mathcal{D}_*^{\text{der}}(\mathcal{E}), \\ \mathcal{D}_*^{\text{der}}(\mathcal{E}) &= \text{reflection of } t_{\leq 2}\pi^{\text{tot}}(\text{ner } D^b(S.\mathcal{E})^{\text{iso}}). \end{aligned}$$

Using the presentations defining $\mathcal{D}_*(C^b(\mathcal{E}))$ and $\mathcal{D}_*^{\text{der}}(\mathcal{E})$ we obtain

Theorem (M.'07)

There is a natural isomorphism $\mathcal{D}_*(C^b(\mathcal{E})) \cong \mathcal{D}_*^{\text{der}}(\mathcal{E})$, *so*

$$\det{}^{\text{der}}(\mathcal{E}, -) \cong \det(C^b(\mathcal{E}), -) : \mathbf{PG}_{\simeq} \longrightarrow \mathbf{Set}.$$

Corollary (Maltsiniotis's first conjecture for K_1)

There are natural isomorphisms

$$K_0(\mathcal{E}) \cong K_0(\mathbb{D}(\mathcal{E})), \qquad K_1(\mathcal{E}) \cong K_1(\mathbb{D}(\mathcal{E})).$$

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The End

Thanks for your attention!

Example

Let $X = \operatorname{Spec} R$ be the spectrum of a local ring R. The graded determinant line bundle

$$\det\colon \mathbf{vect}(X)^{\mathrm{iso}} \longrightarrow \mathbf{Pic}(X)$$

induces an isomorphism in $\mathbf{PG}_{\simeq}$

$$V(\mathbf{vect}(X)) \xrightarrow{\cong} \mathbf{Pic}(X),$$

and moreover these two Picard groupoids are isomorphic in $\mathbf{PG}_{\simeq}$ to the loop Picard groupoid of the following stable quadratic module,

$$\begin{array}{c} \mathbb{Z} \otimes \mathbb{Z} \xrightarrow{\langle \cdot, \cdot \rangle} R^* \xrightarrow{0} \mathbb{Z}, \\ \langle m, n \rangle \quad = \quad (-1)^{mn}. \end{array}$$

◂ back