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The derived Calabi–Yau Auslander–Iyama correspondence

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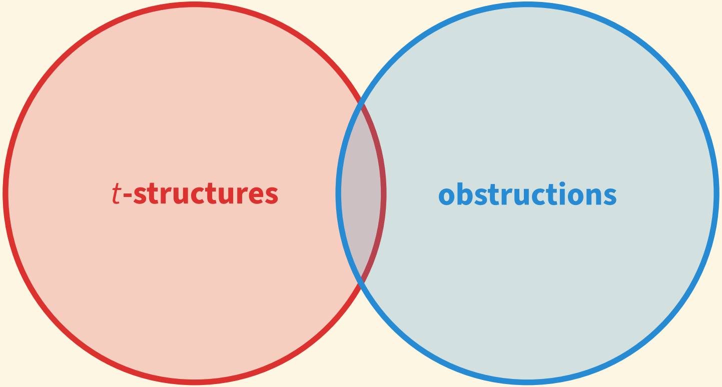
Uniqueness of enhancements of triangulated categories



Lunts, Orlov, Canonaco, Stellari, Neeman

Jasso, Muro

The derived Auslander correspondence

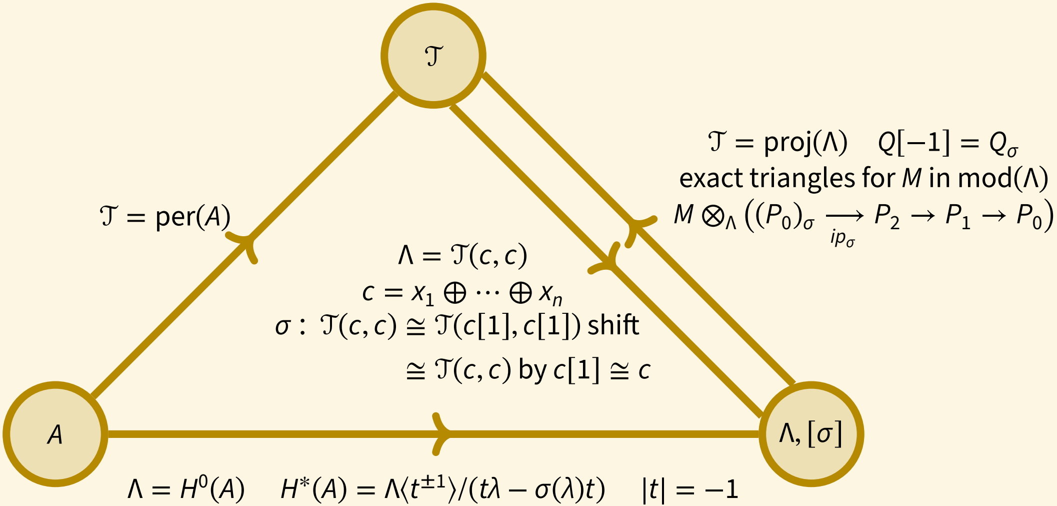
Theorem (Muro 2022)

Over a perfect field k, there is a bijection between equivalence classes of:

1. A a **differential graded algebra (DGA)** such that:
   a. $H^0(A)$ is a finite-dimensional basic algebra,
   b. $A \in \mathrm{per}(A)$ is an additive generator.
2. $\mathcal{T}$ an idempotent-complete, hom-finite **triangulated category** with finitely-many indecomposables $x_1, \dots, x_n$ up to isomorphism.
3. $(\Lambda, [\sigma])$ where:
   a. Λ is a **self-injective basic algebra**,
   b. $[\sigma] \in \mathrm{Out}(\Lambda)$ such that Λ is σ**-twisted 3-periodic**, i.e. there is an exact sequence of Λ-bimodules

$$\Lambda_\sigma \overset{i}{\hookrightarrow} \underbrace{P_2 \to P_1 \to P_0}_{\text{projective-injective}} \overset{p}{\twoheadrightarrow} \Lambda.$$

The derived Auslander correspondence

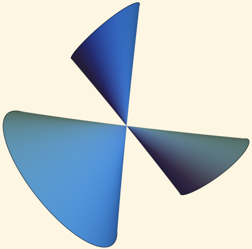


The derived Auslander correspondence

Example

The following $\mathcal{T}$ fit in the derived Auslander correspondence:

1. $\mathcal{T} = \underline{\mathrm{mod}}(B)$ for B a finite-dimensional self-injective representation finite algebra.
2. $\mathcal{T} = D^{sg}(R) = D^b(R)/\operatorname{per}(R)$ for $\operatorname{Spec}(R)$ a complete simple surface singularity.



Cluster tilting

If $\mathcal{T}$ is an idempotent-complete, hom-finite triangulated category, an object c in $\mathcal{T}$ is ***$d\mathbb{Z}$-cluster tilting*** for some $d \geq 1$ if:

$$\begin{aligned} c[d] &\cong c, \\ \operatorname{add}(c) &= \{x \in \mathcal{T} \mid \mathcal{T}(x, c[i]) = 0 \ \forall 0 < i < d\} \\ &= \{x \in \mathcal{T} \mid \mathcal{T}(c, x[i]) = 0 \ \forall 0 < i < d\}. \end{aligned}$$

For $d = 1$, an object is $1\mathbb{Z}$-cluster tilting $\Leftrightarrow$ it is an additive generator.

The derived Auslander–Iyama correspondence

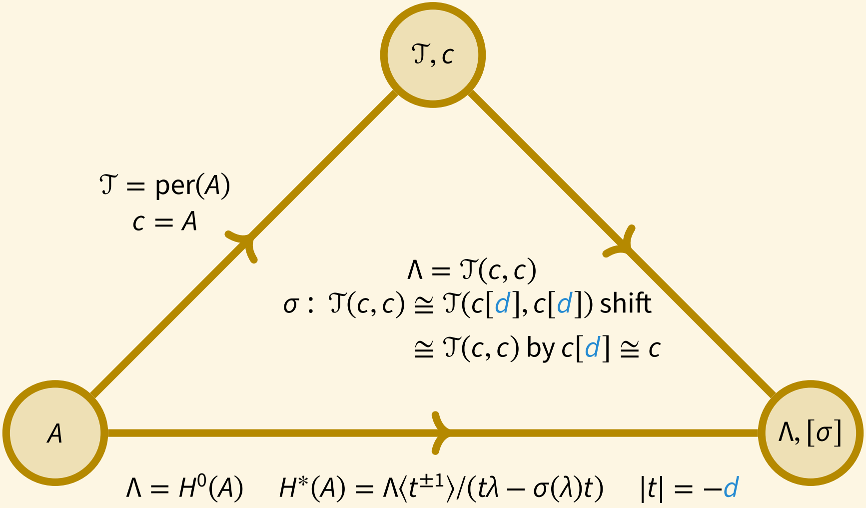
Theorem (Jasso and Muro 2023)

Over a perfect field k, for $d \geq 1$, there is a bijection between equivalence classes of:

1. A a **differential graded algebra** such that:
   a. $H^0(A)$ is a finite-dimensional basic algebra,
   b. $A \in \mathrm{per}(A)$ is $d\mathbb{Z}$-cluster tilting.
2. $(\mathcal{T}, c)$ were:
   a. $\mathcal{T}$ is an idempotent-complete, hom-finite, **algebraic triangulated category**,
   b. c is a basic $d\mathbb{Z}$-cluster tilting object in $\mathcal{T}$.
3. $(\Lambda, [\sigma])$ where:
   a. Λ is a **self-injective basic algebra**,
   b. $[\sigma] \in \mathrm{Out}(\Lambda)$ such that Λ is σ**-twisted (**d**+2)-periodic**, i.e. there is an exact sequence of Λ-bimodules

$$\Lambda_\sigma \hookrightarrow \underbrace{P_{d+1} \to \cdots \to P_0}_{\text{projective-injective}} \twoheadrightarrow \Lambda.$$

The derived Auslander–Iyama correspondence

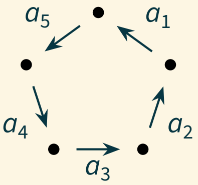


The derived Auslander–Iyama correspondence

Example

The following $(\mathcal{T}, c)$ fit in the derived Auslander–Iyama correspondence:

1. $\mathcal{T} = D^{sg}(R)$ for $\operatorname{Spec}(R)$ a complete cDV singularity with a crepant resolution ($d = 2$), Donovan and Wemyss 2016, Keller's appendix in Jasso and Muro 2023.
2. $\mathcal{T} = \mathcal{C}(\Gamma) = \operatorname{per}(\Gamma)/\operatorname{fd}(\Gamma)$, $c = [\Gamma]$, the Amiot–Guo–Keller cluster category of the following DGAs:
   a. $\Gamma = \mathbf{\Pi}_{d+1}(B)$ the derived preprojective algebra of a d-representation finite algebra B. Here $\Lambda = \Pi_{d+1}(B)$ is the ordinary preprojective algebra.
   b. $\Gamma = \Gamma(Q, W)$ the Ginzburg DGA of a quiver Q with potential W with finite-dimensional self-injective Jacobian algebra $\Lambda = J(Q, W)$ ($d = 2$), Amiot 2009, Keller and Liu 2024.

$Q =$ 

$W = a_1 \cdots a_5$

The bimodule Calabi–Yau property

Let A be a DGA. The ***dual*** $DM = \hom(M, k)$ of an A-bimodule M is an A-bimodule with

$$(a \cdot f \cdot b)(m) = (-1)^{|a|(|f|+|b|+|m|)} f(b \cdot m \cdot a).$$

The DGA A is ***bimodule n-Calabi–Yau*** if there is a quasi-isomorphism of A-bimodules

$$A[n] \xrightarrow{\sim} DA.$$

The DGAs associated to the previous examples via the derived Auslander–Iyama correspondence are bimodule d-Calabi–Yau.

> ***Question***
> How does the bimodule n-Calabi–Yau property fit into the derived Auslander–Iyama correspondence?

Calabi–Yau triangulated categories and the Nakayama automorphism

A triangulated category $\mathcal{T}$ is ***n-Calabi–Yau*** if the n-fold shift functor is a graded Serre functor, i.e. there are natural isomorphisms compatible with the shift

$$\mathcal{T}(x, y[n]) \cong D\mathcal{T}(y, x).$$

If A is a bimodule n-Calabi–Yau DGA $\Rightarrow$ per(A) is n-Calabi–Yau.

Given a pair $(\mathcal{T}, c)$ where:

- $\mathcal{T}$ is idempotent-complete d-Calabi–Yau,
- c is a basic $d\mathbb{Z}$-cluster tilting object in $\mathcal{T}$,

for some $d \geq 1$, then $\Lambda = \mathcal{T}(c, c)$ is ν-twisted $(d+2)$-periodic for $\nu \in \mathrm{Aut}(\Lambda)$ the ***Nakayama automorphism***,

$${}_{\nu}\Lambda \cong D\Lambda.$$

Hochschild cohomology

Given a graded algebra A, the **Hochschild cohomology**

$$\mathrm{HH}^{\star,*}(A) = \mathrm{Ext}^{\star,*}_{A^e}(A, A)$$

is a **Gerstenhaber algebra**, i.e.

- a graded **commutative** algebra,
- a graded **Lie** algebra shifted by -1,

and both structures are compatible in the following way

$$[x, y \cdot z] = [x, y] \cdot z + (-1)^{(|x|-1)|y|} y \cdot [x, z].$$

Twisted Laurent polynomial algebras and Massey products

Let Λ be a σ-twisted $(d+2)$-periodic self-injective basic algebra,

$$\eta : \Lambda_\sigma \hookrightarrow \underbrace{P_{d+1} \to \cdots \to P_0}_{\text{projective-injective}} \twoheadrightarrow \Lambda,$$

The ***d-sparse σ-twisted Laurent polynomial algebra*** is

$$\Lambda(\sigma, d) = \frac{\Lambda\langle t^{\pm 1}\rangle}{(t\lambda - \sigma(\lambda)t)}, \qquad |\Lambda| = 0, \qquad |t| = -d.$$

The inclusion of the degree 0 part $\Lambda \subset \Lambda(\sigma, d)$ induces a map such that

$$\begin{aligned} \mathrm{HH}^{d+2,-d}(\Lambda(\sigma,d)) &\longrightarrow \mathrm{HH}^{d+2}(\Lambda,\Lambda_\sigma) = \mathrm{Ext}^{d+2}_{\Lambda^e}(\Lambda,\Lambda_\sigma), \\ \exists!\ m &\longmapsto [\eta], \end{aligned}$$

with $\frac{1}{2}[m,m] = 0$. We call m a ***universal Massey product*** **(UMP)** of $(\Lambda, [\sigma])$.

The derived Calabi–Yau Auslander–Iyama correspondence

If Λ is a self-injective basic algebra with **Nakayama automorphism** $\nu \in \mathrm{Aut}(\Lambda)$ then $\mathrm{HH}^{\star,*}(\Lambda(\nu, d))$ is equipped with a degree -1 **Batalin–Vilkovisky operator** Δ such that

$$[x, y] = \Delta(x \cdot y) - \Delta(x) \cdot y - (-1)^{|x|} x \cdot \Delta(y).$$

Theorem (Jasso and Muro 2025b)

Over a perfect field k, for $d \geq 1$, there is a bijection between equivalence classes of:

1. A a **differential graded algebra** such that:
   a. $A \in \mathrm{per}(A)$ is basic $d\mathbb{Z}$-cluster tilting,
   b. A is bimodule d-Calabi–Yau.
2. Λ a **self-injective basic algebra** such that:
   a. Λ is ν-twisted $(d+2)$-periodic,
   b. $(\Lambda, [\nu])$ has a universal Massey product $m \in \mathrm{HH}^{d+2,-d}(\Lambda(\nu, d))$ such that

$$\Delta(m) = 0.$$

More Massey products

A DGA $\rightsquigarrow (H^*A, m_3, m_4, \dots)$ A_∞-algebra **minimal model.**

$$m_3(a_1, a_2, a_3) \in \langle a_1, a_2, a_3 \rangle, \quad a_i \in H^*A, \quad \textbf{Masey product}.$$

The ***universal Massey product***[1] **(UMP)** of a DGA A with d-sparse cohomology

$$m_A = \{m_{d+2}\} \in \mathrm{HH}^{d+2,-d}(H^*A), \qquad \frac{1}{2}[m_A, m_A] = 0.$$

Under the derived Auslander–Iyama correspondence, the following notions agree:

- The universal Massey product $m_A \in \mathrm{HH}^{d+2,-d}(H^*A)$ of A.
- The universal Massey product $m_{\Lambda,[\sigma]} \in \mathrm{HH}^{d+2,-d}(\Lambda(\sigma, d))$ of $(\Lambda, [\sigma])$.

[1] Baues and Dreckmann 1989; Benson, Krause, and Schwede 2004; Kaledin 2007…

Formality-like results for algebras

Theorem (Kadeishvili 1988)

If A is a DGA with $\mathrm{HH}^{n+2,-n}(H^*A) = 0$, $n > 0$, then any other DGA B with $H^*B = H^*A$ is $B \simeq A$.

A d-**Massey algebra** (A, m) is a d-sparse graded algebra A and

$$m \in \mathrm{HH}^{d+2,-d}(A), \qquad \frac{1}{2}[m, m] = 0.$$

Its **Hochschild cohomology** $\mathrm{HH}^{\star,*}(A, m)$ is the cohomology of

$$\big(\,\mathrm{HH}^{\star,*}(A)\,,\; d = [m, -]\big).$$

Theorem (Jasso and Muro 2023)

If A is a DGA with d-sparse cohomology and $\mathrm{HH}^{n+2,-n}(H^*A, m_A) = 0$, $n > d$, then any other DGA B with d-sparse cohomology and $(H^*B, m_B) = (H^*A, m_A)$ is $B \simeq A$.

Bimodule Hochschild cohomology

Given a graded algebra A and an A-bimodule M, we define a ***bimodule Hochschild cohomology*** of this pair

$$\mathrm{HH}^{\star,*}(A|M).$$

It is a Gerstenhaber algebra. Moreover, it fits into a long exact sequence

$$\cdots \to \mathrm{HH}^{p,q}(A|M) \to \mathrm{HH}^{p,q}(A) \xrightarrow{\delta} \mathrm{Ext}^{p,q}_{A^e}(M,M) \to \mathrm{HH}^{p+1,q}(A|M) \to \cdots$$

where

$$\delta(x) = 1_M \cdot x - x \cdot 1_M$$

vanishes iff the $\mathrm{HH}^{\star,*}(A)$-bimodule $\mathrm{Ext}^{\star,*}_{A^e}(M,M)$ is symmetric.

Even more Massey products

A DGA and M an A-bimodule $\rightsquigarrow (H^*M, m^M_{0,2}, m^M_{1,1}, m^M_{2,0}, \dots)$ A_∞-bimodule over $(H^*A, m^A_3, m^A_4, \dots)$ **minimal model**.

$$m^M_{1,1}(a,m,b) \in \langle a,m,b \rangle, \quad m \in H^*M, \quad a,b \in H^*A, \quad \textbf{Masey product}.$$

The ***universal Massey product*** **(UMP)** of a bimodule M over a DGA A, both with d-sparse cohomology,

$$m_M = \{m^A_{d+2} + \textstyle\sum_{p+q=d+1} m^M_{p,q}\} \in \mathrm{HH}^{d+2,-d}(H^*A|H^*M), \qquad \frac{1}{2}[m_M, m_M] = 0.$$

It satisfies

$$\begin{aligned} \mathrm{HH}^{d+2,-d}(H^*A|H^*M) &\longrightarrow \mathrm{HH}^{d+2,-d}(H^*A), \\ m_M &\longmapsto m_A. \end{aligned}$$

Formality-like results for bimodules

Theorem (Jasso and Muro 2025a)

If A is a DGA and M is an A-bimodule with $\mathrm{Ext}^{n+1,-n}_{H^*(A)^e}(H^*M, H^*M) = 0$, $n > 0$, then any other A-bimodule N with $H^*N = H^*M$ is $N \simeq M$.

A d-**Massey bimodule** (M, m_M) over a d-Massey algebra (A, m_A) consists of a d-sparse A-bimodule M and a bimodule Hochschild cohomology class with

$$\begin{aligned} \mathrm{HH}^{d+2,-d}(A|M) &\longrightarrow \mathrm{HH}^{d+2,-d}(A), \\ m_M &\longmapsto m_A, \end{aligned}$$

and $\frac{1}{2}[m_M, m_M] = 0$. Its **Hochschild cohomology** $\mathrm{HH}^{\star,*}(M, m_M)$ is the cohomology of

$$\left(\mathrm{Ext}^{\star,*}_{A^e}(M, M),\ d = [m_M, -] \right).$$

Formality-like results for bimodules

Theorem (Jasso and Muro 2025a)

If M is a bimodule over a DGA A, both with d-sparse cohomology, such that the $\mathrm{HH}^{\star,*}(H^*A)$-module $\mathrm{Ext}^{\star,*}_{H^*(A)^e}(H^*M, H^*M)$ is symmetric and $\mathrm{HH}^{n+1,-n}(H^*M, m_M) = 0$ for $n > d$, then any other A-bimodule N with $(H^*N, m_N) = (H^*M, m_M)$ is $N \simeq M$.

If A is a graded algebra and $M = A$, the bimodule Hochschild cohomology is

$$\mathrm{HH}^{\star,*}(A|A) = \mathrm{HH}^{\star,*}(A)[\varepsilon]/(\varepsilon^2) = \mathrm{HH}^{\star,*}(A) \oplus \mathrm{HH}^{\star,*}(A) \cdot \varepsilon, \qquad |\varepsilon| = 1.$$

If A is a DGA with d-sparse cohomology, the UMP of $M = A$ is

$$m_A + 0 \cdot \varepsilon \in \mathrm{HH}^{\star,*}(H^*A)[\varepsilon]/(\varepsilon^2).$$

Moreover, if H^*A is d-Calabi–Yau, $H^*A \cong D(H^*A)[-d]$, then the UMP of $M = DA[-d]$ is

$$m_A + \Delta(m_A) \cdot \varepsilon \in \mathrm{HH}^{\star,*}(H^*A)[\varepsilon]/(\varepsilon^2).$$

THE END

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