Representability of cohomology theories

Fernando Muro

Universidad de Sevilla
Departamento de Álgebra

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Cohomology

Space $=$ pointed simplicial (or cell) complex.

Definition

A reduced cohomology theory H is:

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\textit{Topology} &\to \textit{Algebra}\\
\textit{space } X &\mapsto H^n(X) \textit{ abelian group}, \quad n \in \mathbb{Z},\\
\textit{map } X \xrightarrow{f} Y &\mapsto H^n(X) \xleftarrow{H^n(f)} H^n(Y) \textit{ homomorphism},\\
\textit{homotopic } f \simeq g \colon X \to Y &\mapsto H^n(f) = H^n(g) \textit{ equal},\\
\textit{base point union } \coprod_{i \in I} X_i &\mapsto \prod_{i \in I} H^n(X_i) \textit{ product},
\end{aligned}
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cofiber sequence *exact sequence*

$$X \to Y \to C_f \to \Sigma X \mapsto H^n(X) \leftarrow H^n(Y) \leftarrow H^n(C_f) \leftarrow H^n(\Sigma X),$$

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$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{q} \Sigma X$$

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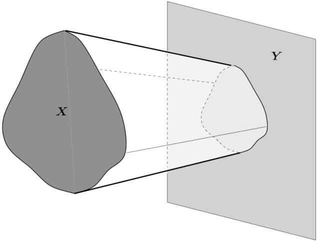

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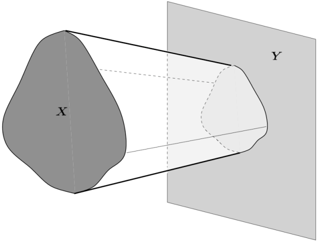

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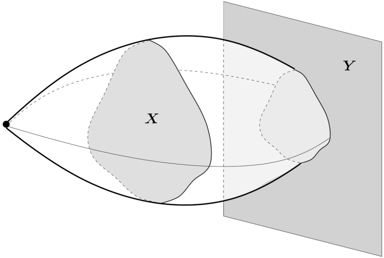

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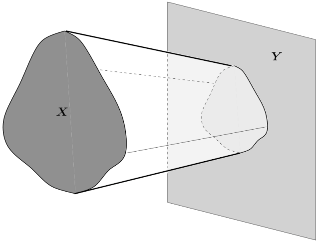

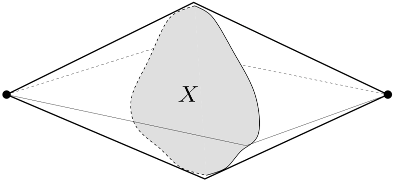

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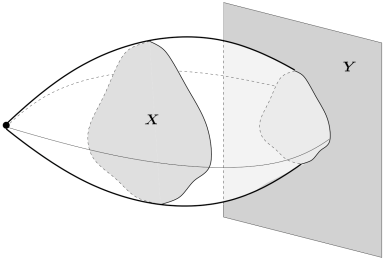

Examples of cohomologies

Example

- *Singular cohomology* $H^*(X,\mathbb{Z})$*, defined on all spaces,*

$$H^n(\textit{discrete}) = 0, \ \textit{for } n \neq 0,$$
$$H^0(X,\mathbb{Z}) = \textit{pointed maps } X \to \mathbb{Z}.$$

▸ course

- *Complex* K*-theory* $K^*(X)$*,* X *compact,*

$$K^0(X) = \textit{stable isomorphism classes of } \mathbb{C}\textit{-vector bundles}/X,$$
$$K^n(X) \cong K^{n+2}(X), \ n \in \mathbb{Z}, \ \textit{Bott periodicity},$$
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Definition

A spectrum $E = \{E_0, E_1, \dots, E_n, \dots\}$ *is a sequence of spaces together with bonding maps,*

$$\Sigma E_n \longrightarrow E_{n+1}, \quad n \geq 0.$$

An Ω*-spectrum is a spectrum* E *where the adjoints of the bonding maps* $E_n \to \Omega E_{n+1}$ *are homotopy equivalences.*

Example

Any space X *defines a spectrum* $\Sigma^\infty X = \{X, \Sigma X, \dots, \Sigma^n X, \dots\}$*, which is not an* Ω*-spectrum.*

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Theorem (G. W. Whitehead'62)

A spectrum E represents a cohomology theory H defined on compact spaces by

$$H^n(X) = \lim_{k\to\infty} [\Sigma^{k-n}X, E_k], \quad n \in \mathbb{Z},$$

where $[-,-]$ denotes the set of homotopy classes of maps.

If E is an Ω-spectrum then E represents a cohomology theory H defined on all spaces by the formula above.

Spectra and cohomology

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Theorem (E. H. Brown'63)

Any cohomology theory defined on all spaces is represented by a spectrum.

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The following corollary can be applied to complex K-theory.

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$E = \{\mathbb{Z}, S^1, \mathbb{C}P^\infty, \dots\}$ with $H^*(X) = H^*(X, \mathbb{Z})$.

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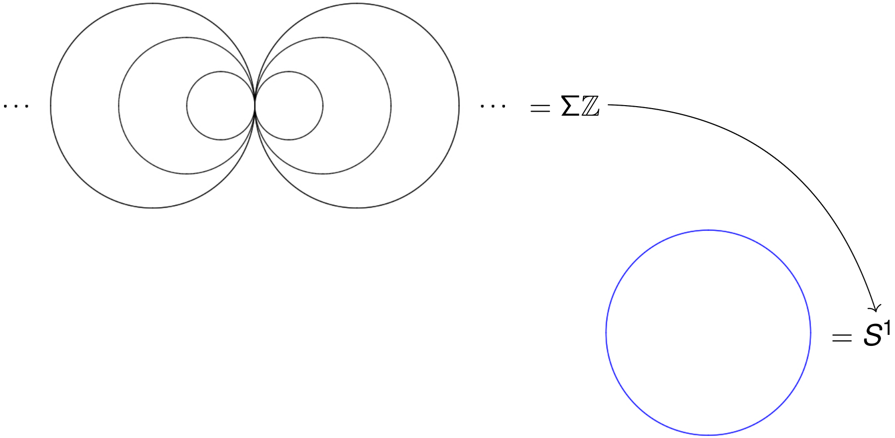

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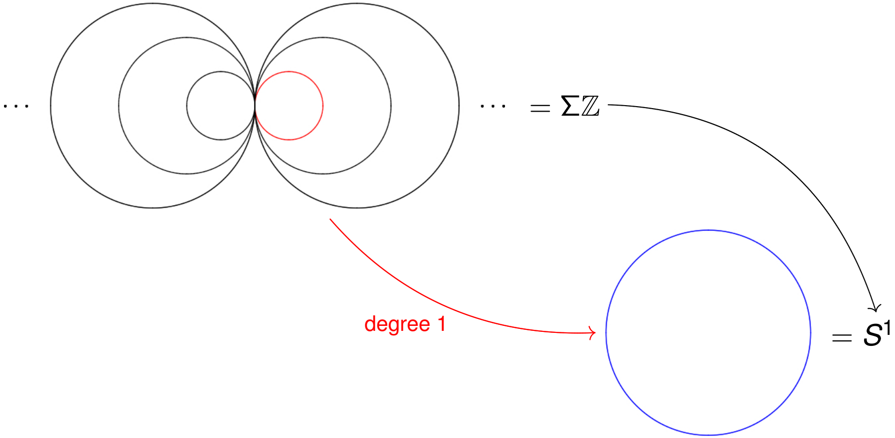

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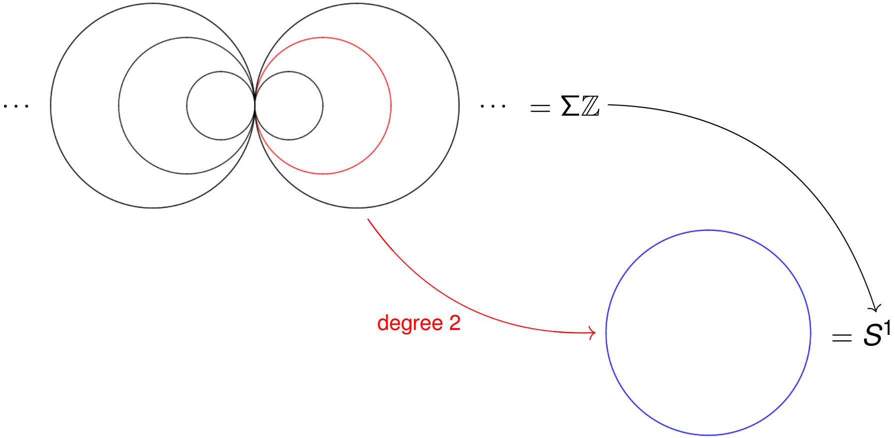

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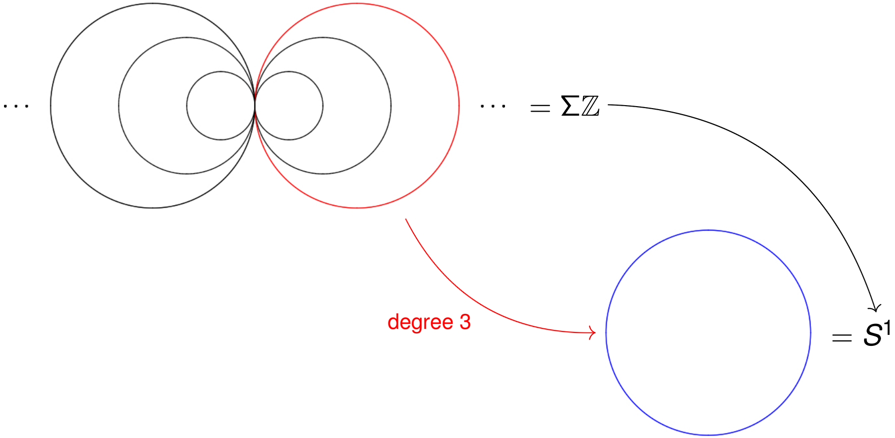

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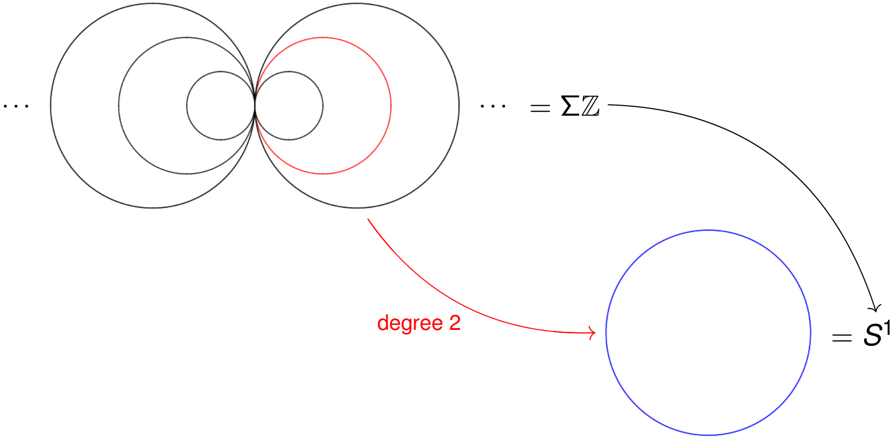

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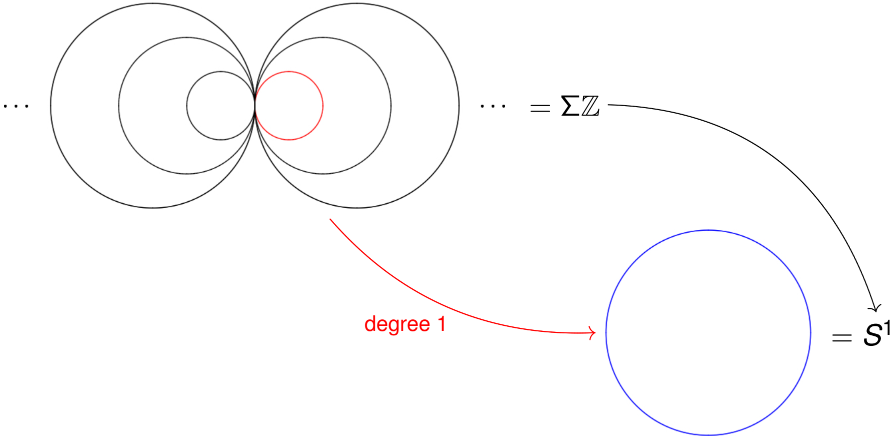

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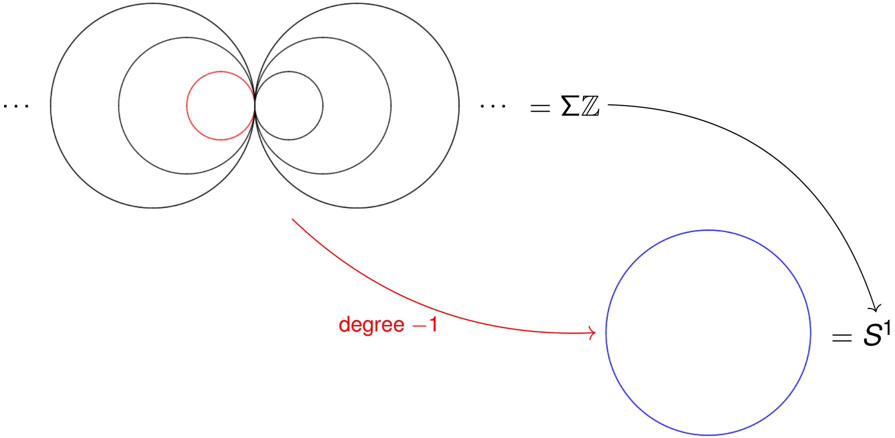

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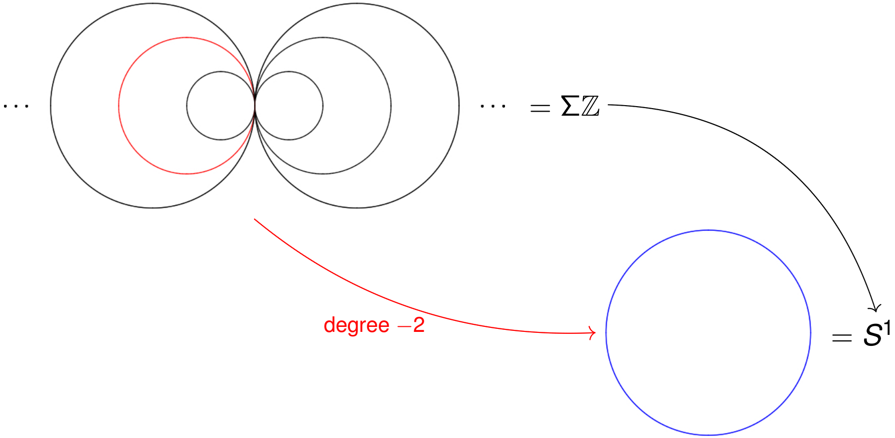

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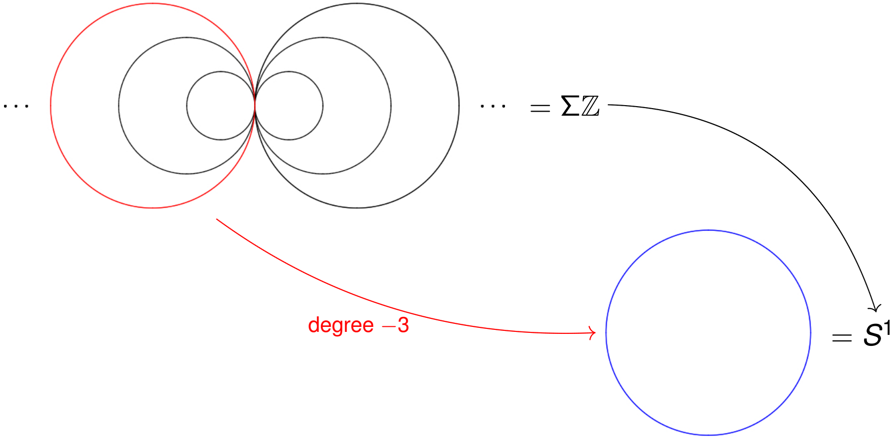

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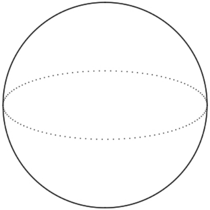
$\Sigma S^1 = S^2 =$  $= \mathbb{C}P^1 \xrightarrow{\text{inclusion}} \mathbb{C}P^\infty$

Stable homotopy

Recall that cohomology regards suspension as an invertible functor:

$$H^n(X) \cong H^{n+1}(\Sigma X).$$

Definition

The compact stable homotopy category $\mathbf{SH}^c$*:*

- *Objects:* (X, n), *X compact space,* $n \in \mathbb{Z}$,

$$(X, n) \sim \Sigma^n X.$$

- *Morphisms:* $\operatorname{Hom}((X, n), (Y, m)) = \lim_{k \to \infty} [\Sigma^{k+n} X, \Sigma^{k+m} Y]$.
- *Suspension:* $\Sigma(X, n) = (X, n+1) \cong (\Sigma X, n)$.
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Definition

A *triangulated category* *consists of:*

- *an additive category* $\mathbf{T}$,
- *an equivalence* $\Sigma\colon \mathbf{T} \xrightarrow{\sim} \mathbf{T}$,
- *a family of* *exact triangles* $X \xrightarrow{f} Y \to C_f \to \Sigma X$ *in* $\mathbf{T}$,

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \arrow[dl] \\ & C_f \arrow[ul, "+1"] & \end{tikzcd}$$

satisfying the formal properties of the compact stable homotopy category.

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Example

Let R be a ring.

- *The homotopy category* $\mathbf{K}(R)$*, objects are complexes of* R*-modules,*

$$C = \cdots \to C_{n+1} \xrightarrow{d_C} C_n \xrightarrow{d_C} C_{n-1} \to \cdots, \quad d_C^2 = 0,$$

$$(\Sigma C)_n = C_{n-1}, \quad d_{\Sigma C} = -d_C,$$

morphisms are chain homotopy classes of maps, and exact triangles come from cofiber sequences of complexes.

- *The derived category* $\mathbf{D}(R) \subset \mathbf{K}(R)$ *is the full subcategory spanned by 'injective resolutions' of complexes.*
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Compactness in triangulated categories

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A triangulated category $\mathbf{T}$ *is cocomplete if it has all coproducts,*

$$\coprod_{i \in I} X_i, \quad I \text{ a set.}$$

An object Y *in* $\mathbf{T}$ *is compact if any map*

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- $\mathbf{D}(\mathbf{A})$ *and* $\mathbf{K}(\mathbf{A})$ *are cocomplete.*
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- *The compact stable homotopy category* $\mathbf{SH}^c$ *admits a cocompletion* $\mathbf{SH}$ *called the full stable homotopy category, whose objects are spectra.*
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A *cohomology theory* *H in a cocomplete triangulated category* $\mathbf{T}$ *is an additive functor*

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A cocomplete triangulated category $\mathbf{T}$ *is compactly generated if any non-trivial object* X *in* $\mathbf{T}$ *admits a non-trivial map* $Y \to X$ *from a compact object* Y *in* $\mathbf{T}^c$.

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- *The stable homotopy category* $\mathbf{SH}$.
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Any cohomology theory $H\colon \mathbf{T}^{op} \to \mathbf{Ab}$ on a compactly generated triangulated category $\mathbf{T}$ is representable.

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Let $\mathbf{T}$ be a compactly generated triangulated category with $\mathbf{T}^c$ countable.

- *Any cohomology theory $H\colon (\mathbf{T}^c)^{op} \to \mathbf{Ab}$ is represented by a not necessarily compact object E in $\mathbf{T}$, $H = \mathrm{Hom}_{\mathbf{T}}(-, E)_{|\mathbf{T}^c}$.*
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This theorem applies to $\mathbf{SH}$, but it also applies to $\mathbf{D}(\mathbb{Z})$.

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Example (Casacuberta–Neeman'09)

Let R be a 'polynomial ring with a proper class of indeterminates', the subcategory of acyclic complexes in $\mathbf{K}(R)$ does not satisfy Brown representability.

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- *Whether Adams representability holds in the derived category of the complex affine plane $\mathbf{D}(\mathrm{Qcoh}/\mathbb{A}^2_{\mathbb{C}})$ depends essentially on the continuum hypothesis.*
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A right $\mathbf{T}^c$-module A is an additive functor $A\colon (\mathbf{T}^c)^{op} \to \mathbf{Ab}$. The category $\mathbf{Mod}(\mathbf{T}^c)$ of right $\mathbf{T}^c$-modules is a Grothendieck abelian category with a set of small projective generators.

Theorem (Neeman'97, Beligiannis'00)

If $\mathbf{T}$ is compactly generated and $H\colon (\mathbf{T}^c)^{op} \to \mathbf{Ab}$ is a cohomology theory then:

- proj dim $H \leq 2 \Rightarrow H$ *is representable.*
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Purity

Purity is the relative homological algebra in $\mathbf{Mod}(R)$ obtained by regarding all finitely presented R-modules as projectives.

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Remark (Baer–Brune–Lenzing'82)

If R is a finite-dimensional hereditary algebra over an uncountable algebraically closed field, then $\mathbf{D}(R)$ satisfies the Adams representability theorem $\Leftrightarrow$ has finite representation type.

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If Y is in $\mathbf{T}^\alpha$ then any map

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A cocomplete triangulated category $\mathbf{T}$ *is* α*-compactly generated if any non-trivial object* X *in* $\mathbf{T}$ *admits a non-trivial map* $Y \to X$ *from an* α*-compact object* Y *in* $\mathbf{T}^{\alpha}$*. A cocomplete triangulated category is well generated if it is* α*-compactly generated for some regular cardinal* α*.*

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- *All compactly generated triangulated categories are well generated.*
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Any cohomology theory $H\colon \mathbf{T}^{op} \to \mathbf{Ab}$ on a well generated triangulated category $\mathbf{T}$ is representable.

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A cohomology theory H for α-compact objects is an additive functor

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What about α-Adams representability?

Conjecture (Rosický'05, Neeman'09)

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Almost nothing is known about $\Rightarrow$ for $\alpha > \aleph_0$.

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A right continuous $\mathbf{T}^\alpha$-module A is an additive functor $A\colon (\mathbf{T}^\alpha)^{op} \to \mathbf{Ab}$ with

$$A(\coprod_{i\in I} X_i) \cong \prod_{i\in I} A(X_i), \quad \text{card } I < \alpha.$$

The category $\mathbf{Mod}_\alpha(\mathbf{T}^\alpha)$ of right continuous $\mathbf{T}^\alpha$-modules is an abelian category but not Grothendienck.

Theorem (M-Raventós'09)

If $\mathbf{T}$ is α-compactly generated and $H\colon (\mathbf{T}^\alpha)^{op} \to \mathbf{Ab}$ is a cohomology theory then:

- proj dim $H \leq 2 \Rightarrow$ *H is representable.*
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A homological criterion for Adams representability

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If R is a finite-dimensional wild hereditary k-algebra, $\operatorname{card} k \geq \aleph_\omega$, then the α-Adams representability theorem is false for all $\alpha < \aleph_\omega$.

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If the conjecture is true, then for any ring R there exists a cardinal α such that for any R-module A there is a short exact sequence,

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such that the modules P_i and Q_j have $< \alpha$ generators and relations.

Moreover, if B is an R-module with $< \alpha$ generators and relations, then

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Under the continuum hypothesis, the following theorem can be applied to $\mathbf{D}(\text{sheaves}/M)$.

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Representability of cohomology theories

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Cofiber sequence of complexes

$$
\begin{tikzcd}
X \arrow[r, "f"] & Y \\
\vdots \arrow[d] & \vdots \arrow[d] \\
X_{n+1} \arrow[r, "f"] \arrow[d, "d_X"] & Y_{n+1} \arrow[d, "d_Y"] \\
X_n \arrow[r, "f"] \arrow[d, "d_X"] & Y_n \arrow[d, "d_Y"] \\
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X \arrow[r, "f"] & Y & & C_f \\
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X_{n+1} \arrow[r, "f"] \arrow[d, "d_X"] & Y_{n+1} \arrow[d, "d_Y"] & & Y_{n+1} \oplus X_n \arrow[d, "\left(\begin{smallmatrix} d_Y & f \\ 0 & -d_X \end{smallmatrix}\right) = d_{C_f}"] \\
X_n \arrow[r, "f"] \arrow[d, "d_X"] & Y_n \arrow[d, "d_Y"] & & Y_n \oplus X_{n-1} \arrow[d, "\left(\begin{smallmatrix} d_Y & f \\ 0 & -d_X \end{smallmatrix}\right) = d_{C_f}"] \\
X_{n-1} \arrow[r, "f"] \arrow[d] & Y_{n-1} \arrow[d] & & Y_{n-1} \oplus X_{n-2} \arrow[d] \\
\vdots & \vdots & & \vdots
\end{tikzcd}
$$

mapping cone

Cofiber sequence of complexes

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "i"] & C_f \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
X_{n+1} \arrow[r, "f"] \arrow[d, "d_X"] & Y_{n+1} \arrow[r, "\text{inclusion}"] \arrow[d, "d_Y"] & Y_{n+1} \oplus X_n \arrow[d, "\left(\begin{smallmatrix} d_Y & f \\ 0 & -d_X \end{smallmatrix}\right) = d_{C_f}"] \\
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$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "i"] & C_f \text{ mapping cone} & \Sigma X \text{ suspension} \\
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\begin{tikzcd}
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\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
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X_{n-1} \arrow[r, "f"] \arrow[d] & Y_{n-1} \arrow[r, "\text{inclusion}"] \arrow[d] & Y_{n-1} \oplus X_{n-2} \arrow[r, "\text{projection"] \arrow[d] & X_{n-2} \arrow[d] \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}
$$

mapping cone

suspension

back

Crash course on singular cohomology

Compute the cohomology of spheres,

$$S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \cdots + x_n^2 = 1\};$$

$$S^0 = \{\pm 1\} = \bullet \qquad \bullet\,, \quad H^0(S^0, \mathbb{Z}) = \text{pointed maps } S^0 \to \mathbb{Z} \cong \mathbb{Z};$$

$\Sigma S^0 =$ $\cong$ $= S^1$,

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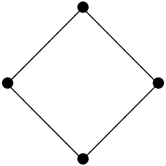
$\Sigma S^0 \quad =$  $\cong$ $= \quad S^1,$

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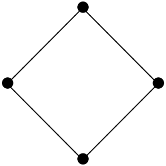
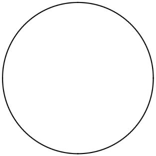
$\Sigma S^0 \quad =$  $\cong$  $= \quad S^1,$

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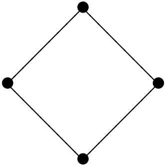
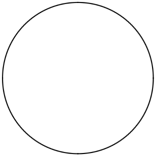
$\Sigma S^0 \quad =$  $\cong$  $= \quad S^1,$

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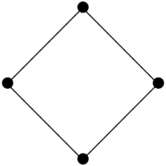
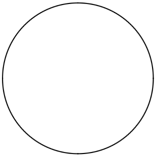
$\Sigma S^0 =$  $\cong$  $= S^1,$

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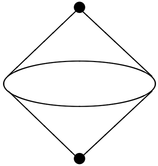

$\cong$

$= S^2,$

$$H^2(S^2, \mathbb{Z}) \cong H^1(S^1, \mathbb{Z}) \cong \mathbb{Z}, \quad H^n(S^2, \mathbb{Z}) = 0 \text{ for } n \neq 2;$$

and inductively,

$$\Sigma S^{n-1} = S^n, \qquad H^n(S^n, \mathbb{Z}) \cong H^{n-1}(S^{n-1}, \mathbb{Z}) \cong \mathbb{Z},$$
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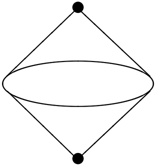
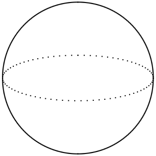
$\Sigma S^1 =$  $\cong$  $= S^2$,

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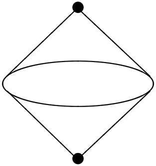
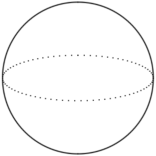
$\Sigma S^1 =$  $\cong$  $= S^2,$

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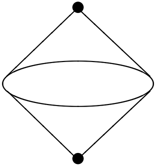
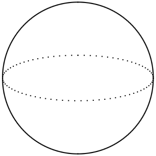
$\Sigma S^1 \quad =$  $\cong$  $= \quad S^2,$

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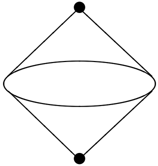
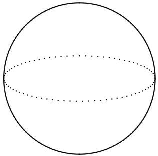
$\Sigma S^1 =$  $\cong$  $= S^2,$

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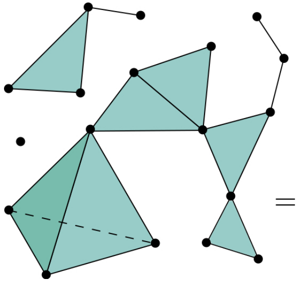
Crash course on singular cohomology

Let C_f be a simplicial complex,



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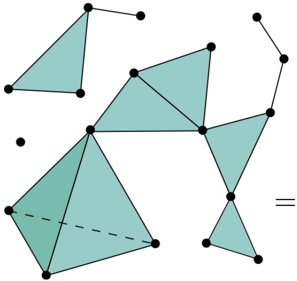

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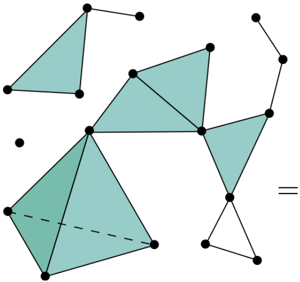
Any simplex is a cone over its boundary, therefore C_f can be obtained from Y,



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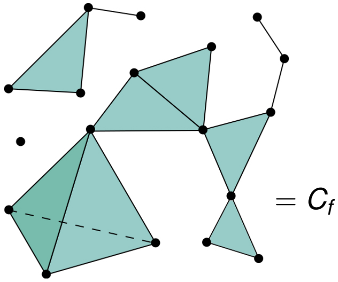

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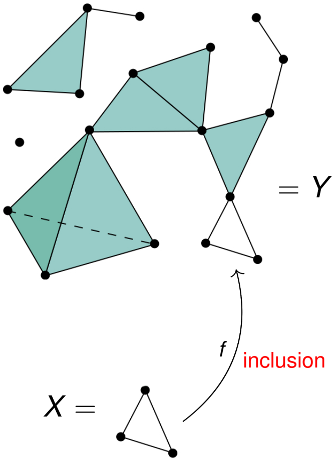

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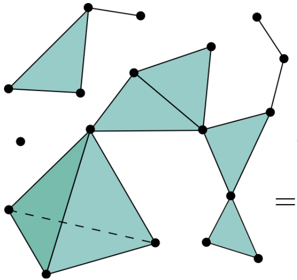

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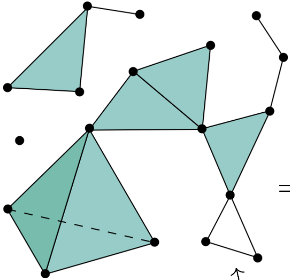

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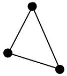
$$S^1 = X \xrightarrow[\text{inclusion}]{f} Y \xrightarrow[\text{inclusion}]{i} C_f \xrightarrow{q} \Sigma X = S^2.$$

$X =$



Given a cofiber sequence

$$S^n \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{q} S^{n+1};$$

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$$0 \leftarrow H^{n+1}(C_f,\mathbb{Z}) \longleftarrow \mathbb{Z} \xleftarrow{H^n(f,\mathbb{Z})} H^n(Y,\mathbb{Z}) \longleftarrow H^n(C_f,\mathbb{Z}) \leftarrow 0,$$

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