Triangulated Categories in Algebra and Geometry

Charles University and Czech Academy of Sciences

Prague, 23 – 27 September 2024

Cluster tilting differential graded algebras

Fernando Muro
Universidad de Sevilla
https://personal.us.es/fmuro/

UNIVERSIDAD
Ð SEVILLA
1505

Table of contents

1. Cluster tilting objects

UNIVERSIDAD
Ð SEVILLA
1505

k perfect ground field

$\mathcal{T}$ small Hom-finite triangulated category with split idempotents

$\Rightarrow$ Krull-Schmidt

$c \in \mathcal{T}$ an object

$\langle c \rangle \subset \mathcal{T}$ smallest thick subcategory containing c

$\operatorname{add}(c) \subset \mathcal{T}$ smallest full subcategory closed under direct sums and summands containing c

$c \in \mathcal{T}$ is basic if $c = x_1 \oplus \cdots \oplus x_n$ indecomposables $x_i \not\cong x_j$ for $i \neq j$

$c \in \mathcal{T}$ is d-cluster tilting, for some $d \geq 1$, if it is basic and

$$\operatorname{add}(c) = \{ x \in \mathcal{T} \mid \mathcal{T}(x, c[i]) = 0 \ \forall\, 0 < i < d \}$$
$$= \{ x \in \mathcal{T} \mid \mathcal{T}(c, x[i]) = 0 \ \forall\, 0 < i < d \}$$

$\Rightarrow \langle c \rangle = \mathcal{T}$

and it is periodic if $c \cong c[d]$

$d=1$ $c \in \mathcal{J}$ is 1-cluster tilting iff it is basic and $\mathcal{J} = \operatorname{add}(c)$

Periodicity is automatic.

additively finite | basic additive generator

unique up to $\cong$

A DGA

$D(A)$ derived category of right A-modules

$\cup$

$D^c(A) = \langle A \rangle$ compact objects

A is (periodic) d-cluster tilting if $D^c(A)$ is Hom-finite and $A \in D^c(A)$ is (periodic) d-cluster tilting.

$\boxed{\text{GOAL understand these DGAs}}$

$\mathcal{T}$ is algebraic if $\mathcal{T} \simeq D^c(A)$ for some DGA A called enhancement

any $c \in \mathcal{T}$ has a derived endomorphism DGA $\mathbb{R}\mathrm{End}(c)$ with

$$H^n \mathbb{R}\mathrm{End}(c) = \mathcal{T}(c, c[n]), \quad n \in \mathbb{Z},$$

$$H^0 \mathbb{R}\mathrm{End}(c) = \mathrm{End}(c).$$

Example if $\mathcal{T}$ is algebraic and $c \in \mathcal{T}$ is (periodic) d-cluster tilting
$\Rightarrow$ the endomorphism DGA $A := \mathbb{R}\mathrm{End}(c)$ of c is (periodic) d-cluster
tilting and $D^c(A) \simeq \mathcal{T}$.

Examples (geometry)

$\mathcal{T} = \underline{\mathrm{CM}}(R)$ stable category of maximal Cohen-Macaulay modules over a Cohen-Macaulay ring R

$\mathcal{T} = \underline{\mathrm{CM}}(R) \simeq D^b(\mathrm{mod}\, R) / \langle R \rangle =: D^{\mathrm{sing}}(R)$ derived category of singularities

1. $\mathcal{T}$ is additively finite if $\operatorname{Spec} R$ is a simple hypersurface singularity $/\mathbb{C}$

[Buchweitz-Greuel-Schreyer '87] $d=1$ e.g. $R = \mathbb{C}[[x,y]]/(x^2-y^3)$ cusp

2. $\operatorname{Spec} R$ isolated compound Du Val singularity with a crepant resolution $/\mathbb{C}$

$\{\text{minimal models } X \to \operatorname{Spec} R\}_{/\cong} \cong \{c \in \mathcal{T} \text{ periodic } 2\text{-cluster tilting}\}_{/\cong}$

[Wemyss '18] e.g. $R = \mathbb{C}[[u,v,x,y]]/(uv+x^2-xy^3)$ Reid's pagoda

Examples (algebra) [Keller'08, Amiot'09, Guo'11]

$\mathcal{C}(\Gamma) := D^c(\Gamma) / \{M \mid \dim H^* M < \infty\}$ **cluster category**

$\overset{\psi}{\Gamma}$ periodic d-cluster tilting for:

1. $\Gamma = \Pi_{d+1}(A)$ derived $(d+1)$-preprojective DGA of
   A d-representation finite f.d. algebra ($\operatorname{gl}\dim A = d$)
   $H^0(\Gamma) = \Pi_{d+1}(A)$ $(d+1)$-preprojective algebra
   [Iyama-Oppermann'13]

2. $\Gamma = \Gamma(Q, W)$ completed Ginzburg DGA of a quiver Q with potential W
   with self-injective completed Jacobian algebra $J(Q,W) = H^0(\Gamma)$ $d=2$
   [Keller-Yang'11, Herschend-Iyama'11]

2. Structure theorem for periodic cluster tilting DGAs

Universidad de Sevilla
1505

Theorem [Jasso-M'23] There is a bijection between:

1. A periodic d-cluster tilting DGA up to quasi-isomorphism
2. (Λ, I) where:
   a) Λ self-injective f.d. basic algebra
   b) I invertible Λ-bimodule with $\Omega^{d+2}_{\Lambda^e}(\Lambda) \cong I$ in $\underline{\mathrm{mod}}(\Lambda^e)$

up to isomorphism of (algebra, bimodule) pairs

$$A \longmapsto (H^0 A, H^{-d} A)$$

Λ basic algebra

$\mathrm{Pic}(\Lambda) := \{\text{invertible } \Lambda\text{-bimodules}\}/_{\cong}$

$\mathrm{Out}(\Lambda) := \mathrm{Aut}(\Lambda)/\mathrm{Inn}(\Lambda) \qquad \mathrm{Inn}(\Lambda) = \mathrm{im}[\Lambda^{\times} \xrightarrow{\text{conj.}} \mathrm{Aut}(\Lambda)]$

$\mathrm{Out}(\Lambda) \cong \mathrm{Pic}(\Lambda) : \sigma \mapsto {}_1\Lambda_\sigma$ twisted bimodule

so (Λ, I) in 2. is equivalent to (Λ, σ)

A periodic d-cluster tilting

$$\Lambda := H^0 A = D^c(A)(A,A) \circlearrowleft \sigma \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"', "\cong"] \arrow[dr, phantom, "\#"] & A \arrow[d, "\cong"', "\alpha"] \\ A[d] \arrow[r, "\sigma(f)"'] & A[d] \end{tikzcd} \qquad H^{-d}A = D^c(A)(A[d],A) \overset{\alpha^*}{\cong} {}_1D^c(A)(A,A)_\sigma = {}_1\Lambda_\sigma$$

Λ self-injective, it is twisted $(d+2)$-periodic if $\Omega^{d+2}_{\Lambda^e}\Lambda \cong {}_1\Lambda_\sigma$ in $\underline{\mathrm{mod}}\,\Lambda^e$

Lemma Λ connected non-separable twisted $(d+2)$-periodic

$\Rightarrow \Omega^{d+2}_{\Lambda^e}\Lambda \cong {}_1\Lambda_\sigma$ in $\mathrm{mod}\,\Lambda^e$ so $\sigma \in \mathrm{Out}(\Lambda)$ is unique

Lemma Λ separable $\Rightarrow \Omega^{d+2}_{\Lambda^e}\Lambda \cong {}_1\Lambda_\sigma$ in $\underline{\mathrm{mod}}\,\Lambda^e = 0 \ \forall\, \sigma \in \mathrm{Out}(\Lambda)$

3. (d+2)-angulated categories

[Geiss - Keller - Oppermann '13]

A $(d+2)$-angulated category is an additive category $\mathcal{C}$ equipped with $\Sigma : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ and a class of diagrams called $(d+2)$-angles

$$X_1 \to X_2 \to X_3 \to \cdots \to X_{d+1} \to X_{d+2} \to \Sigma X_1$$

satisfying:

$$\forall X \quad X \to X \to 0 \to \cdots \to 0 \to 0 \to \Sigma X$$

$$X_1 \xrightarrow{\forall f} X_2 \xrightarrow{\exists \dots} X_3 \to \cdots \to X_{d+1} \to X_{d+2} \to \Sigma X_1$$

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d, "g"'] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d, "\exists \dots"'] & \cdots \arrow[r] & X_{d+1} \arrow[r] \arrow[d] & X_{d+2} \arrow[r] \arrow[d] & \Sigma X_1 \arrow[d, "\Sigma g"] \\
Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3 \arrow[r] & \cdots \arrow[r] & Y_{d+1} \arrow[r] & Y_{d+2} \arrow[r] & \Sigma Y_1
\end{tikzcd}$$

+ analogues of rotation and octahedral axioms

$d=1 \equiv$ triangulated

Example [Geiss-Keller-Oppermann '13]

$\mathcal{C} = \operatorname{add}(c) \subset \mathcal{T}$ with $c \in \mathcal{T}$ periodic d-cluster tilting, $\Sigma = [d]$

a $(d+2)$-angle in $\mathcal{C}$ is

$$\begin{tikzcd}
 & X_2 \arrow[rr] \arrow[dr] \arrow[d, phantom, "\text{exact}"] & & X_3 \arrow[dr] \arrow[d, phantom, "\text{exact}"] \arrow[rrrr, phantom, "\cdots\cdots\cdots"] & & & & X_{d+1} \arrow[dr] \arrow[d, phantom, "\text{exact}"] & \\
X_1 \arrow[ur] & & X_{2.5} \arrow[ll, "+1"] \arrow[ur] \arrow[u, phantom, "\#"] & & X_{3.5} \arrow[ll, "+1"] & \cdots \arrow[l] & X_{d.5} \arrow[l] \arrow[ur] & & X_{d+2} \arrow[ll, "+1"] \arrow[llllllll, bend left, "+d"] \arrow[llll, phantom, "\#"]
\end{tikzcd}$$

d exact triangles in $\mathcal{T}$

$\in \mathcal{C}$

$\Lambda := \operatorname{End}(c) \to \operatorname{add}(c) \simeq \operatorname{proj} \Lambda \to \Lambda$ self-injective

Theorem [Jasso-M'23] $c \in \mathcal{T}$ periodic d-cluster tilting $\iff$ $\mathcal{C} = \operatorname{add}(c)$, $\Sigma = [d]$ and the previous $(d+2)$-angles form a $(d+2)$-angulated category.

Example [Lin'19] after [Amiot'07] for $d=1$

Λ f.d. basic self-injective algebra

$$\underbrace{{}_1\Lambda_\sigma \overset{i}{\hookrightarrow} P_{d+2} \to \cdots \to P_2 \to P_1}_{\in \operatorname{proj} \Lambda^e} \overset{p}{\twoheadrightarrow} \Lambda$$ $(d+2)$-extension of Λ-bimodules

$\leftrightarrow$ twisted $(d+2)$-periodic

$\mathcal{C} = \operatorname{proj} \Lambda$ $\qquad$ $\Sigma^{-1} = - \otimes_\Lambda {}_1\Lambda_\sigma$

$(d+2)$-angles $\quad M \otimes_\Lambda (P_1 \otimes_\Lambda {}_1\Lambda_\sigma \to P_{d+2} \to \cdots \to P_2 \to P_1)$ $\quad M \in \operatorname{mod} \Lambda$

$P \otimes_\Lambda {}_1\Lambda_\sigma \to {}_1\Lambda_\sigma \overset{i}{\to} P_{d+2}$

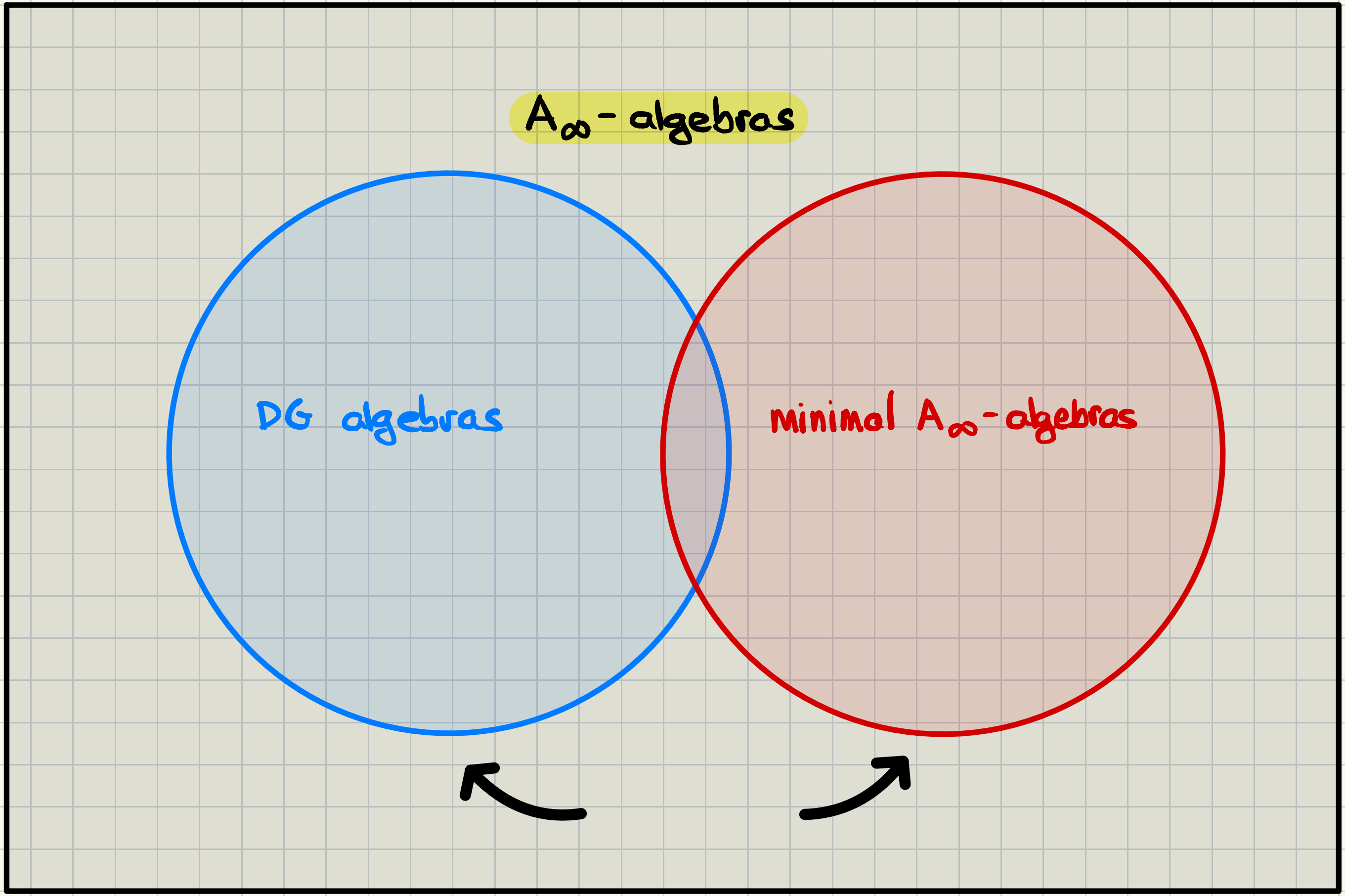
Theorem [Jasso-M'23] These $(d+2)$-angles can be defined for

$${}_1\Lambda_\sigma \hookrightarrow P_{d+2} \to \underbrace{\cdots \to P_2 \to P_1}_{\in \operatorname{proj} \Lambda^e} \twoheadrightarrow \Lambda.$$ They yield a $(d+2)$-angulated category

$\Leftrightarrow P_{d+2} \in \operatorname{proj} \Lambda^e$

4. A-infinity algebras

UNIVERSIDAD
Ð SEVILLA
1505


A_∞-algebras
DG algebras
minimal A_∞-algebras

An A_∞-algebra is a graded vector space B equipped with operations

$$m_n : B \otimes \overset{n}{\cdots} \otimes B \longrightarrow B \quad , \quad n \geq 1,$$

of degree $2-n$ such that:

$$\sum_{p+q=n+1} \pm\, m_p(\ldots, m_q, \ldots) = 0 \quad \forall n \geq 1$$

1. m_1 is a differential, $m_1^2 = 0$
2. m_1 satisfies the Leibniz rule w.r.t. the binary product m_2
3. m_2 is associative up to the cochain homotopy m_3

⋮ ...

DGA $\leftrightarrow$ $m_n = 0 \ \forall\, n \geq 3$

minimal $\leftrightarrow$ $m_1 = 0$ $\rightarrow$ B is a graded associative algebra with product m_2

$\rightarrow$ $m_n \in C^{n,2-n}(B,B)$ is a Hochschild cochain, $n \geq 3$

4. $\rightarrow$ m_3 is a Hochschild cocycle $\rightarrow$ $\{m_3\} \in HH^{3,-1}(B,B)$

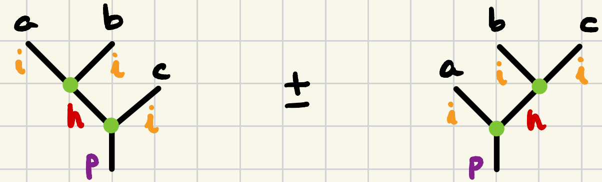
[Baues-Dreckmann '89, Benson-Krause-Schwede '04 ...] universal Massey product

Minimal model of a DGA A. Underlying graded algebra H^*A. Choose:

$$B = H^*A \underset{p}{\overset{i}{\rightleftarrows}} A \circlearrowleft h$$

with i cocycle selection graded vector space morphism, $pi = id_{H^*A}$ retraction, $h : ip \simeq id_A$ cochain homotopy + side conditions. Define the higher operations:

$m_3(a,b,c) =$



$$p(h(i(a) \cdot i(b)) \cdot i(c)) \qquad p(i(a) \cdot h(i(b) \cdot i(c)))$$

m_n = sum indexed by planar binary trees with n leaves, $n \geq 3$.

A periodic d-cluster tilting DGA $\longrightarrow$ H^*A concentrated in degrees $d\mathbb{Z}$ (*)

$\rightarrow$ $m_n = 0$ in the minimal model for $n \geq 3$ unless $d \mid (n-2)$

$\rightarrow$ $m_{d+2} \in C^{d+2,-d}(H^*A, H^*A)$ Hochschild cocycle

$\rightarrow$ $\{m_{d+2}\} \in HH^{d+2,-d}(H^*A, H^*A)$ universal Massey product of length $d+2$

$$\Big\downarrow j^* \qquad\qquad j : H^0A \hookrightarrow H^*A$$

$j^*\{m_{d+2}\} \in HH^{d+2,-d}(H^0A, H^*A)$ restricted universal Massey product of length $d+2$

$$\begin{aligned} &= HH^{d+2}(H^0A, H^{-d}A) \\ &= HH^{d+2}(\Lambda, {}_1\Lambda_\sigma) \\ &= \mathrm{Ext}^{d+2}_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma) \end{aligned}$$

$\Lambda = H^0A$ $\qquad$ $H^{-d}A \cong {}_1\Lambda_\sigma$

$A \cong A[di]$

(*) Actually $H^*A = \bigoplus_{di \in d\mathbb{Z}} {}_1\Lambda_{\sigma^{-i}} = \dfrac{\Lambda\langle t^{\pm 1}\rangle}{(\lambda t - t\sigma(\lambda))_{\lambda\in\Lambda}} = \Lambda(\sigma, d)$ $\qquad$ $|t| = -d$

5. Recognition theorem for periodic cluster tilting DGAs

UNIVERSIDAD Ð SEVILLA
1505

Theorem [Jasso-M'23] TFAE for A DGA

1. A is a periodic d-cluster tilting DGA
2. The following properties hold:
   a) $\Lambda := H^0 A$ is a self-injective f.d. basic algebra
   b) $H^* A$ is concentrated in $d\mathbb{Z}$
   c) $H^* A \cong H^* A[d]$ as $H^* A$-modules $(\to H^{-d} A = {}_1\Lambda_\sigma)$
   d) $j^* \{m_{d+2}\} \in \mathrm{Ext}^{d+2}_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma)$ is represented by an extension ${}_1\Lambda_\sigma \hookrightarrow \underbrace{P_{d+2} \to \cdots \to P_2 \to P_1}_{\in \operatorname{proj} \Lambda^e} \twoheadrightarrow \Lambda$

Moreover, if they hold then the [Geiss-Keller-Oppermann'13] $(d+2)$-angulated structure on $\operatorname{proj} \Lambda \simeq \operatorname{add}(A) \subset D^c(A)$ coincides with [Lin'19]'s.

d) $\Leftrightarrow$ $j^*\{m_{d+2}\} \in \underline{HH}^{*,*}(H^0A, H^*A)$, Hochschild-Tate cohomology, is a bidegree $(d+2,-d)$ unit.

self-injective

$j^*\{m_{d+2}\} = 0 \Leftrightarrow \Lambda = H^0A$ separable $\Rightarrow A$ is almost never formal

Example R compound Du Val singularity $/\mathbb{C}$

$A = \mathbb{R}\mathrm{End}(c)$ for $c \in \underline{CM}(R)$ periodic 2-cluster tilting

A formal $\Leftrightarrow R \cong \mathbb{C}[[u,v,x,y]]/(uv-xy)$ Atiyah flop

6. Bimodule Calabi-Yau periodic cluster tilting DGAs

UNIVERSIDAD
Ɖ SEVILLA
1505

A graded algebra or DGA

DA dual A-bimodule $A^i = \mathrm{Hom}_k(A^{-i}, k)$

A is bimodule n-Calabi-Yau if $A[n] \cong DA$ in $D(A^e)$

If B is a bimodule n-Calabi-Yau graded algebra

$\Rightarrow HH^{*,*}(B,B)$ is a Batalin-Vilkovisky algebra

i.e. graded commutative + $HH^{*,*}(B,B) \circlearrowleft \Delta$ differential operator of bidegree $(-1,0)$ and order ≤ 2

Theorem [Jasso-M] A periodic d-cluster tilting DGA. TFAE:

1. A is d-Calabi-Yau
2. H^*A is d-Calabi-Yau and $\boxed{\Delta(\{m_{d+2}\}) = 0}$

7. Some open questions

UNIVERSIDAD Ð SEVILLA
1505

Λ f.d. algebra which is
- basic
- connected
- non-separable
- twisted $(d+2)$-periodic

Find explicitly:

1. A periodic d-cluster tilting DGA with $H^0 A = \Lambda$
2. $(\Lambda(\sigma, d), \boxed{m_{d+2}}, m_{2d+2}, \ldots, m_{nd+2}, \ldots)$ minimal model of A (A_∞-algebra)
3. Down-to-earth description of $\mathcal{T} = D^c(A)$
4. d-BCY structure on A or $(\Lambda(\sigma, d), \{m_{nd+2}\}_{n\geq 1})$ when $\Lambda(\sigma, d)$ is d-BCY and $\boxed{\Delta(\{m_{d+2}\}) = 0}$
5. Does this equation always hold? Proof or counterexample

THANKS A LOT