On determinants (as functors)

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Categorification of determinants

From Wikipedia:

"In mathematics, ***categorification*** *refers to the process of replacing set-theoretic theorems by category-theoretic analogues."*

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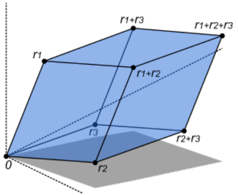
Categorification of determinants

$$n \times n \text{ matrix } M \quad \leftrightsquigarrow \quad f\colon k^n \to k^n \text{ homomorphism}$$

If $k = \mathbb{R}$, $|\det(M)|$ is the scale factor for f.



Let $\omega = e_1 \wedge \cdots \wedge e_n \in \wedge^n k^n$ be the volume form,

$$\begin{aligned} \wedge^n f\colon \wedge^n k^n &\longrightarrow \wedge^n k^n, \\ \omega &\mapsto \det(M)\,\omega. \end{aligned}$$

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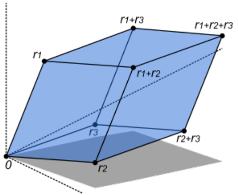

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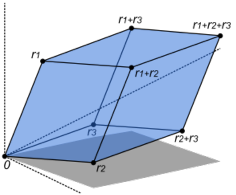

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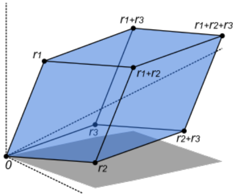

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Categorification of determinants

For any f. d. vector space A and any isomorphism $f\colon A \xrightarrow{\sim} B$ we set

$$\begin{aligned} \det(A) &= (\wedge^{\dim A} A, \dim A), \\ \det(f) &= \wedge^{\dim A} f, \end{aligned}$$

in the category $\mathbf{lines}^{\mathbb{Z}}$ of graded lines:

- Objects (L, n) are given by L a vector space of dim $= 1$ and $n \in \mathbb{Z}$.
- Morphisms $(L, n) \to (L', n')$ are isomorphisms $L \to L'$ if $n = n'$ and $\emptyset$ otherwise.

The functor

$$\det\colon \mathbf{vect}^{\mathrm{iso}} \longrightarrow \mathbf{lines}^{\mathbb{Z}}$$

categorifies determinants.

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Categorification of determinants

The functor det satisfies further properties.

The category **lines**$^{\mathbb{Z}}$ is a Picard groupoid, i.e. a symmetric categorical group, with tensor product

$$(L,n)\otimes(L',n') \quad = \quad (L\otimes L', n+n'),$$

and commutativity constraint twisted by a sign

$$\begin{array}{ccc} (L,n)\otimes(L',n') & \stackrel{\text{comm.}}{\longrightarrow} & (L',n')\otimes(L,n), \\ v\otimes w & \mapsto & (-1)^{nn'}w\otimes v. \end{array}$$

Categorification of determinants

Given a s. e. s.

$$\Delta = A \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} B/A$$

we have an additivity isomorphism

$$\det(\Delta)\colon \det(B/A) \otimes \det(A) \longrightarrow \det(B)$$

defined as follows. Choose bases $\{v_1, \dots, v_p\}$ of B/A and $\{w_1, \dots, w_q\}$ of A, and set

$$(v_1 \wedge \cdots \wedge v_p) \otimes (w_1 \wedge \cdots \wedge w_q) \overset{\det(\Delta)}{\mapsto} v'_1 \wedge \cdots \wedge v'_p \wedge i(w_1) \wedge \cdots \wedge i(w_q),$$

where $p(v'_r) = v_r$.

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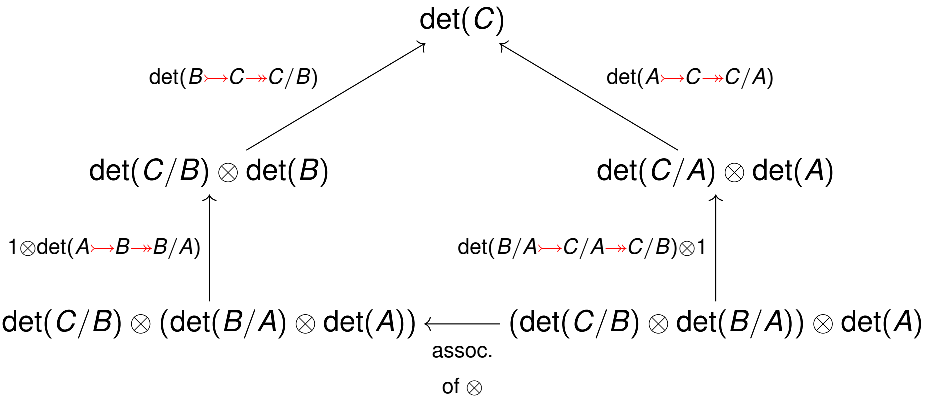
Categorification of determinants

Additivity isomorphisms are natural with respect to s. e. s. isomorphisms,

$$\begin{array}{ccccc} A & \rightarrowtail & B & \twoheadrightarrow & B/A \\ \wr\downarrow f & & \wr\downarrow g & & \wr\downarrow h \\ A' & \rightarrowtail & B' & \twoheadrightarrow & B'/A' \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \det(B/A)\otimes\det(A) & \xrightarrow{\det(\Delta)} & \det(B) \\ {\scriptstyle \det(h)\otimes\det(f)}\downarrow & & \downarrow{\scriptstyle \det(g)} \\ \det(B'/A')\otimes\det(A') & \xrightarrow[\det(\Delta')]{} & \det(B') \end{array}$$

Categorification of determinants

They are associative, i.e. for each 2-step filtration $A \rightarrowtail B \rightarrowtail C$ the following diagram commutes



Categorification of determinants

They are commutative, i.e. the following diagram commutes

$$
\begin{array}{ccc}
 & \det(A \oplus B) & \\
\det(B \rightarrowtail A\oplus B \twoheadrightarrow A) \nearrow & & \nwarrow \det(A \rightarrowtail A\oplus B \twoheadrightarrow B) \\
\det(A) \otimes \det(B) & \xrightarrow[\text{comm. of } \otimes]{} & \det(B) \otimes \det(A)
\end{array}
$$

Determinant for exact categories

What's special about det above?

- **lines**$^{\mathbb{Z}}$ is a Picard groupoid,
- **vect** has short exact sequences.

Definition (Deligne'87)

Let **E** *be an abelian or exact category and* **P** *a Picard groupoid. A determinant is a functor*

$$\det\colon \mathbf{E}^{\text{iso}} \longrightarrow \mathbf{P}$$

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Determinants for exact categories

Example

- $\mathbf{E} = \mathbf{vect}(X)$ *the exact category of vector bundles over* X*,*
- $\mathbf{P} = \mathbf{Pic}(X)$ *the category of graded line bundles* (L, n)*, with* L *a line bundle over* X *and* $n\colon X \to \mathbb{Z}$ *a locally constant map.*

One can define a determinant functor of $\mathbf{vect}(X)$ *with values on* $\mathbf{Pic}(X)$ *as above, by using exterior powers.*

In the special case $X = \mathrm{Spec}(R)$*,* $\mathbf{E} = \mathbf{proj}(R)$ *and* $\mathbf{P} = \mathbf{Pic}(R)$ *is the Picard groupoid of graded projective* R*-modules of constant rank* 1*.*

What if R is noncommutative? Do we have any canonical $\mathbf{P}$ in this case?

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Determinants for exact categories

Definition

A *natural isomorphism* *between determinant functors is a natural isomorphism*

$$\tau\colon \det \Rightarrow \det'\colon \mathbf{E}^{\mathrm{iso}} \longrightarrow \mathbf{P},$$

such that for any s. e. s. $\Delta = A \rightarrowtail B \twoheadrightarrow B/A$ *the following diagram commutes*

$$\begin{array}{ccc} \det(B/A)\otimes\det(A) & \xrightarrow{\det(\Delta)} & \det(B) \\ {\scriptstyle \tau(A)\otimes\tau(B/A)}\downarrow & & \downarrow{\scriptstyle \tau(B)} \\ \det'(B/A)\otimes\det'(A) & \xrightarrow[\det'(\Delta)]{} & \det'(B) \end{array}$$

Determinant functors and natural iso. form a groupoid $\det(\mathbf{E},\mathbf{P})$.

Determinants for exact categories

Theorem (Deligne'87)

The 2-functor

$$\det(\mathbf{E}, -)\colon \mathbf{PicGrd} \longrightarrow \mathbf{Grd}$$

is representable.

A representing Picard groupoid $V(\mathbf{E})$ is called a category of virtual objects.

Example

$V(\mathbf{proj}(R)) \simeq \mathbf{Pic}(R)$ *if the commutative ring R is local, semisimple, or the ring of integers in a number field.*

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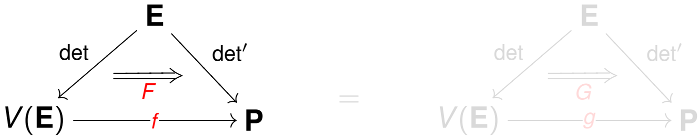
The category of virtual objects comes equipped with a universal determinant functor

$$\det\colon \mathbf{E}^{\mathrm{iso}} \longrightarrow V(\mathbf{E})$$

such that any other determinant functor $\det'\colon \mathbf{E} \to \mathbf{P}$ factorises as



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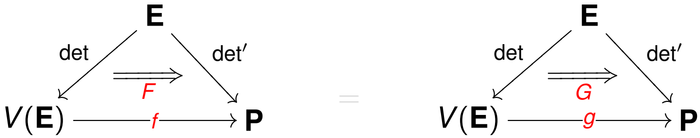

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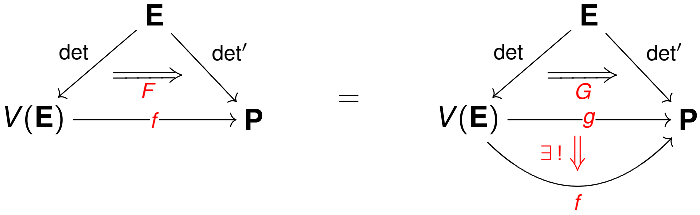

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The homotopy groups of a Picard groupoid $\mathbf{P}$ are

$$\begin{aligned} \pi_0\mathbf{P} &= \text{isomorphism classes of objects, the sum is induced by } \otimes, \\ \pi_1\mathbf{P} &= \operatorname{Aut}_{\mathbf{P}}(I), \text{ the automorphisms of the unit object.} \end{aligned}$$

The Postnikov invariant of $\mathbf{P}$ is the homomorphism

$$\pi_0\mathbf{P} \xrightarrow{\eta} \pi_1\mathbf{P},$$

such that

$$\eta(x) \otimes x \otimes x = \text{comm.}: x \otimes x \longrightarrow x \otimes x.$$

Example

$\pi_0\mathbf{Pic}(X) \cong H^0(X, \mathbb{Z}) \oplus H^1(X, \mathcal{O}_X^\times)$ *and* $\pi_1\mathbf{Pic}(X) \cong \mathcal{O}_X^\times(X)$.

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Theorem (Deligne'87)

There are natural isomorphisms

$$\begin{aligned}\pi_0 V(\mathbf{E}) &\cong K_0(\mathbf{E}),\\ \pi_1 V(\mathbf{E}) &\cong K_1(\mathbf{E}),\end{aligned}$$

such that the Postnikov invariant of $V(\mathbf{E})$ corresponds to the action of the stable Hopf map $0 \neq \eta \in \pi_1(S) \cong \mathbb{Z}/2$ on Quillen's K-theory.

Actually Segal's classifying spectrum $B(V(\mathbf{E}))$ is naturally isomorphic to the 1-type of Quillen's K-theory spectrum $K(\mathbf{E})$ in the stable homotopy category.

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Determinants for complexes

Knudsen–Mumford'76 tackled the problem of defining the determinant of a bounded complex A^* in $\mathbf{E} = \mathbf{vect}(X)$,

$$\cdots \to A^{n-1} \xrightarrow{d} A^n \xrightarrow{d} A^{n+1} \to \cdots,$$

$$\det(A^*) \quad = \quad \bigotimes_{n\in\mathbb{Z}} \det(A^n)^{(-1)^n}.$$

However given a quasi-isomorphim $f\colon A^* \xrightarrow{\sim} B^*$ it is not obvious to produce an isomorphism $\det(f)\colon \det(A^*) \to \det(B^*)$, etc...

Determinants for Waldhausen categories

Given an exact category **E**, the category of bounded complexes $C^b(\mathbf{E})$ is a Waldhausen category:

- a weak equivalence is a quasi-isomorphism $f\colon A^* \xrightarrow{\sim} B^*$,
- a cofibration is a levelwise admissible monomorphism $f\colon A^* \rightarrowtail B^*$,
- a cofiber sequence is a levelwise s. e. s. $A^* \rightarrowtail B^* \twoheadrightarrow B^*/A^*$.

Exact categories are also examples of Waldhausen categories, the weak equivalences are the isomorphisms and the cofibrations are the admissible monomorphisms.

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Determinants for Waldhausen categories

Definition (Knudsen'02, M–Tonks–Witte'08)

Let $\mathbf{W}$ *be a Waldhausen category and* $\mathbf{P}$ *a Picard groupoid. A determinant is a functor*

$$\det\colon \mathbf{W}^{\mathrm{we}} \longrightarrow \mathbf{P}$$

together with an additivity isomorphism

$$\det(\Delta)\colon \det(B/A)\otimes\det(A) \longrightarrow \det(B)$$

for each cofiber sequence $\Delta = A \rightarrowtail B \twoheadrightarrow B/A$ *in* $\mathbf{W}$ *satisfying naturality with respect to weak equivalences of cofiber sequences, associativity and commutativity.*

One can similarly define natural isomorphisms between these determinant functors in order to obtain a groupoid $\det(\mathbf{W},\mathbf{P})$.

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Let $V(\mathbf{W})$ be a representative.

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Theorem (M–Tonks–Witte'08)

The 2-functor

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Actually Segal's classifying spectrum $B(V(\mathbf{W}))$ is naturally isomorphic to the 1-type of Waldhausen's K-theory spectrum $K(\mathbf{W})$ in the stable homotopy category.

Corollary (Knudsen–Mumford'76, Knudsen'02)

The inclusion $\mathbf{E} \subset C^b(\mathbf{E})$ *induces a natural equivalence*

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It follows from the Gillet–Waldhausen theorem which asserts that the inclusion induces an isomorphism $K_*(\mathbf{E}) \cong K_*(C^b(\mathbf{E}))$.

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The bounded derived category $D^b(\mathbf{E})$ is obtained from $C^b(\mathbf{E})$ by inverting quasi-isomorphisms, therefore a determinant functor $\det\colon C^b(\mathbf{E})^{\mathrm{we}} \to \mathbf{P}$ induces a functor

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The category $D^b(\mathbf{E})$ is triangulated, it is equipped with exact triangles,

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Determinants for triangulated categories

Definition (Breuning'06)

Let **T** *be a triangulated category and* **P** *a Picard groupoid. A determinant is a functor*

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for each exact triangle $\Delta = A \to B \to C \to A[1]$ *in* **T** *satisfying naturality with respect to triangle isomorphisms, associativity with respect to octahedral diagrams, and commutativity.*

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It follows from Neeman's heart theorem which asserts that the inclusion induces an isomorphism $K_*(\mathbf{A}) \cong K_*(D^b(\mathbf{A}))$. Actually we can replace $D^b(\mathbf{A})$ by any triangulated category $\mathbf{T}$ with a non-degenerate bounded t-structure with heart $\mathbf{A}$.

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This corollary is not true for arbitrary exact categories.

Let $\mathbf{E} = \mathbf{proj}(k[\varepsilon]/(\varepsilon^2))$ be the category of f. g. free modules over the ring of dual numbers. For this exact category,

$$k \hookrightarrow K_1(\mathbf{E}) \overset{\text{incl.}}{\twoheadrightarrow} K_1(D^b(\mathbf{E})) \cong k^\times,$$

the kernel is generated by $\det(1+\varepsilon)$.

Schlichting showed that there is no possible K-theory for triangulated categories satisfying the usual theorems and agreeing with Waldhausen's. This example explicitly shows that Neeman's K-theory of triangulated categories does not satisfy agreement.

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A Grothendieck derivator is a 2-functor

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where $\mathbf{W}$ is a Waldhausen category with cylinders and a saturated class of weak equivalences.

There is a notion of determinant functor for derivators such that $\det(\mathbb{D}(\mathbf{W}), \mathbf{P}) \simeq \det^{\mathrm{der}}(\mathbf{W}, \mathbf{P})$.

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Small models for virtual objects

A stable quadratic module C_* is a diagram

$$C_0^{ab} \otimes C_0^{ab} \xrightarrow{\langle\cdot,\cdot\rangle} C_1 \xrightarrow{\partial} C_0 \quad \text{satisfying} \quad \begin{array}{rcl} \partial\langle c_1, d_1\rangle & = & [d_1, c_1], \\ \langle \partial(c_2), \partial(d_2)\rangle & = & [d_2, c_2], \\ \langle c_1, d_1\rangle & = & -\langle d_1, c_1\rangle. \end{array}$$

The loop Picard groupoid ΩC_* has object set C_0 and morphisms

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The homotopy groups of the loop Picard groupoid ΩC_* are

$$\begin{aligned} \pi_0 \Omega C_* &= C_0/\partial(C_1), \\ \pi_1 \Omega C_* &= \operatorname{Ker} \partial, \end{aligned}$$

and the Postnikov invariant is

$$\begin{aligned} \eta \colon \pi_0 \Omega C_* &\longrightarrow \pi_1 \Omega C_*, \\ x &\mapsto \langle x, x \rangle. \end{aligned}$$

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The category of virtual objects $V(\mathbf{W}) \simeq \Omega\mathcal{D}_*\mathbf{W}$, where $\mathcal{D}_*\mathbf{W}$ is the stable quadratic module generated in dimension zero by the symbols

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and in dimension one by

- $[A \stackrel{\sim}{\rightarrow} A']$ for any weak equivalence,
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- The boundary relations ▸ formulas ▸ bisimplices.
- Composition of weak equivalences ▸ formula ▸ bisimplex.
- Weak equivalences of cofiber sequences ▸ formula ▸ bisimplex.
- Composition of cofiber sequences ▸ formula ▸ bisimplex.
- Coproducts ▸ formula.

▸ skip

Small models for virtual objects

The generating symbols satisfy six kinds of relations, corresponding to the laws of a determinant functor.

- The trivial relations formulas bisimplices.
- The boundary relations formulas bisimplices.
- Composition of weak equivalences formula bisimplex.
- Weak equivalences of cofiber sequences formula bisimplex.
- Composition of cofiber sequences formula bisimplex.
- Coproducts formula.

skip

The trivial relations

- $[*] = 0$.
- $[A \xrightarrow{1_A} A] = 0$.
- $[A \xrightarrow{1_A} A \twoheadrightarrow *] = 0$, $[* \rightarrowtail A \xrightarrow{1_A} A] = 0$.

This proves that the universal det preserves identities. back

The boundary relations

- $\partial[A \xrightarrow{\sim} A'] = -[A'] + [A]$.
- $\partial[A \rightarrowtail B \twoheadrightarrow B/A] = -[B] + [B/A] + [A]$.

This allows to define the universal det as

$$\begin{aligned} \det(A) &= [A], \\ \det(A \xrightarrow{\sim} A') &= ([A'], [A \xrightarrow{\sim} A']), \\ \det(A \rightarrowtail B \twoheadrightarrow B/A) &= ([B], [A \rightarrowtail B \twoheadrightarrow B/A]). \end{aligned}$$

◂ back

Composition of weak equivalences

- For any pair of composable weak equivalences $A \xrightarrow{\sim} A' \xrightarrow{\sim} A''$,

$$[A \xrightarrow{\sim} A''] = [A' \xrightarrow{\sim} A''] + [A \xrightarrow{\sim} A'].$$

This proves that the universal det preserves composition.

back

Weak equivalences of cofiber sequences

- For any commutative diagram in **W** as follows

$$\begin{array}{ccccc} A & \rightarrowtail & B & \twoheadrightarrow & B/A \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ A' & \rightarrowtail & B' & \twoheadrightarrow & B'/A' \end{array}$$

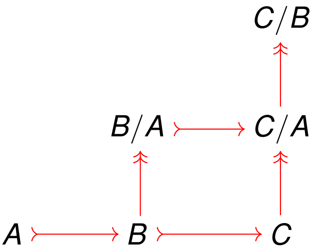
  we have

$$\begin{array}{rcl} [A' \rightarrowtail B' \twoheadrightarrow B'/A'] & & \\ [A \xrightarrow{\sim} A'] + [B/A \xrightarrow{\sim} B'/A'] & & \\ +\langle [A], -[B'/A'] + [B/A] \rangle & = & [B \xrightarrow{\sim} B'] \\ & & +[A \rightarrowtail B \twoheadrightarrow B/A]. \end{array}$$

This proves that additivity isomorphisms are natural. back

Composition of cofiber sequences

- For any commutative diagram consisting of four obvious cofiber sequences in **W** as follows



we have (this implies associativity of additivity isomorphisms)

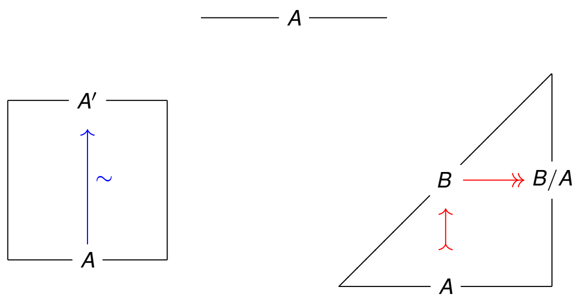
$$\begin{aligned}
&[B \rightarrowtail C \twoheadrightarrow C/B] \\
&+[A \rightarrowtail B \twoheadrightarrow B/A] \quad = \quad [A \rightarrowtail C \twoheadrightarrow C/A] \\
&\qquad\qquad\qquad\qquad\qquad\;\; +[B/A \rightarrowtail C/A \twoheadrightarrow C/B] \\
&\qquad\qquad\qquad\qquad\qquad\;\; +\langle [A], -[C/A] + [C/B] + [B/A] \rangle.
\end{aligned}$$

- For any pair of objects A, B in $\mathbf{W}$

$$\langle [A],[B]\rangle = -[A \overset{i_1}{\rightarrowtail} A\vee B \overset{p_2}{\twoheadrightarrow} B] + [B \overset{i_2}{\rightarrowtail} A\vee B \overset{p_1}{\twoheadrightarrow} A].$$

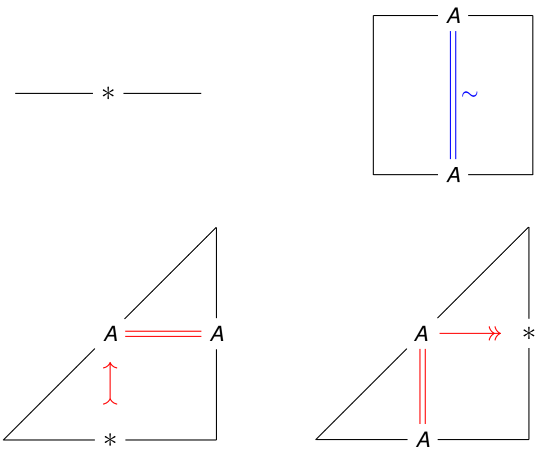
This implies commutativity of additivity isomorphisms. back

Bisimplices of total degree 1 and 2



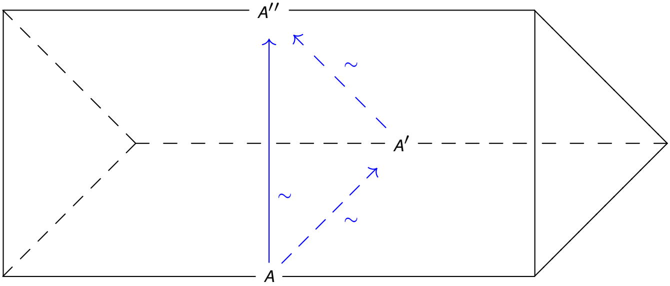
back to generators back to relations

Degenerate bisimplices of total degree 1 and 2



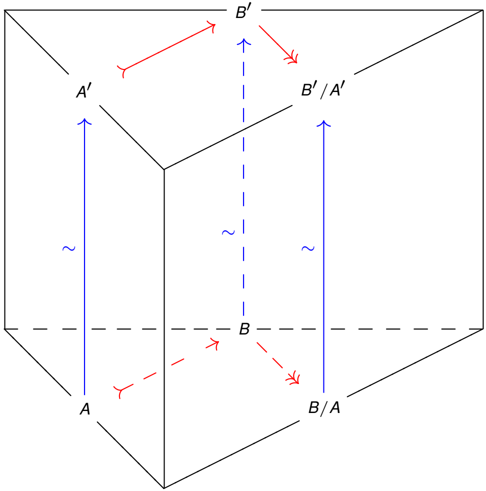
back

Bisimplex of bidegree $(1, 2)$



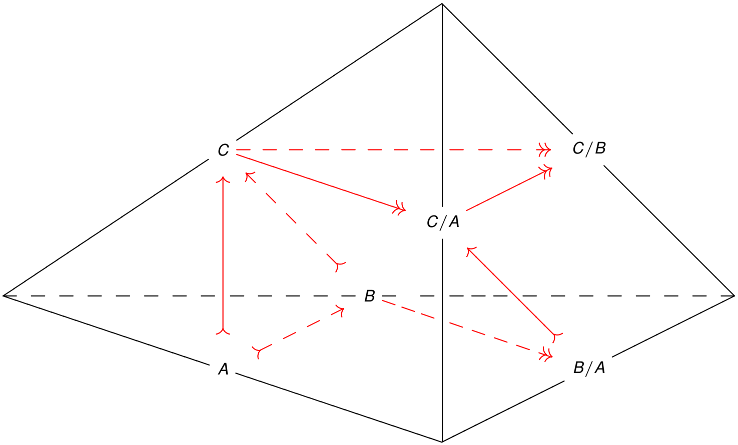
back

Bisimplex of bidegree (2, 1)



back

Bisimplex of bidegree $(3, 0)$



◂ back

The End

Thanks for your attention!