# The triangulated Auslander–Iyama correspondence

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# Main theorem

## Theorem

Let $k$ be a perfect field and $d \geq 1$. There is a bijective correspondence between equivalence classes of:

1. $(\mathcal{T}, c)$ were:
   (a) $\mathcal{T}$ algebraic triangulated category with split idempotents such that $\dim \mathcal{T}(x, y) < \infty$ for all $x, y \in \mathcal{T}$.
   (b) $c \in \mathcal{T}$ basic $d\mathbb{Z}$-cluster tilting object.
2. $(\Lambda, [\sigma])$ where:
   (a) $\Lambda$ twisted $(d+2)$-periodic basic Frobenius algebra.
   (b) $[\sigma] \in \mathrm{Out}(\Lambda)$ such that $\Omega^{d+2}(\Lambda) \cong {}_1\Lambda_\sigma$ in $\underline{\mathrm{mod}}(\Lambda^e)$.
3. Differential graded algebras (DGAs) $A$ such that:
   (a) $\dim H^0(A) < \infty$.
   (b) $A \in D^c(A)$ is a basic $d\mathbb{Z}$-cluster tilting object.

# The equivalence relations

1. $(\mathcal{T}, c) \sim (\mathcal{T}', c')$ if there is a triangulated equivalence

   $$F \colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}'$$

   such that

   $$F(c) \cong c'.$$

2. $(\Lambda, [\sigma]) \sim (\Lambda', [\sigma'])$ if there is an isomorphism

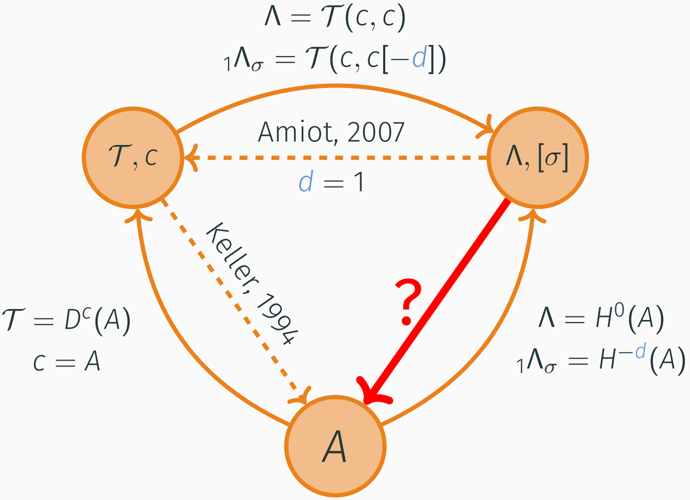
   $$f \colon \Lambda \xrightarrow{\sim} \Lambda'$$

   such that

   $$[\sigma] = [f^{-1}\sigma' f] \in \mathrm{Out}(\Lambda).$$

3. Quasi-isomorphisms.

# The bijections



The twisted $(d+2)$-periodicity of $\Lambda$ is equivalent to the existence of an extension of $\Lambda$-bimodules with projective-injective middle terms $P_i$, $1 \le i \le d+2$,

$$\eta : \qquad {}_1\Lambda_\sigma \hookrightarrow P_{d+2} \to \cdots \to P_1 \twoheadrightarrow \Lambda.$$

This represents

$$\{\eta\} \in \mathrm{Ext}^{d+2}_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma) = HH^{d+2}(\Lambda, {}_1\Lambda_\sigma).$$

The set of all these classes are an orbit for the action of $Z(\Lambda)^\times$.

# Differential graded algebra

$A$ is a DGA such that $\Lambda = H^0(A)$ is finite-dimensional and $A \in D^c(A)$ is basic $d\mathbb{Z}$-cluster tilting. Hence, there exists a unique $[\sigma] \in \mathrm{Out}(\Lambda)$ such that

$$H^{-d}(A) \cong {}_1\Lambda_\sigma$$

and hence

$$\begin{aligned} H^{-di}(A) &\cong {}_1\Lambda_{\sigma^i} && i \in \mathbb{Z}, \\ H^n(A) &= 0 && \text{otherwise.} \end{aligned}$$

Therefore

$$H^*(A) = \Lambda(\sigma, d) = \frac{\Lambda\langle \imath^{\pm 1}\rangle}{(\imath\lambda - \sigma(\lambda)\imath)}, \qquad |\imath| = -d.$$

This algebra is $d$-sparse, i.e. it is concentrated in degrees $d\mathbb{Z}$.

# Minimal models

$A$ has an essentially unique minimal ($A_\infty$-algebra) model, consisting of operations

$$m_{di+2}\colon \Lambda(\sigma, d)\otimes \overset{di+2}{\cdots} \otimes\Lambda(\sigma, d) \longrightarrow \Lambda(\sigma, d), \quad |m_{di+2}| = -di, \quad i \geq 1,$$

satisfying certain equations. These operations are Hochschild cochains

$$m_{di+2} \in C^{di+2,-di}(\Lambda(\sigma, d), \Lambda(\sigma, d)).$$

The first one ($i = 1$) is a cocycle whose cohomology class is called universal Massey product (UMP) of length $d + 2$,

$$\{m_{d+2}\} \in HH^{d+2,-d}(\Lambda(\sigma, d), \Lambda(\sigma, d)).$$

# The connection

$\Lambda \subset \Lambda(\sigma, d)$ is the degree 0 part.

${}_1\Lambda_\sigma \subset \Lambda(\sigma, d)$ is the degree $-d$ part.

The inclusion $j \colon \Lambda \hookrightarrow \Lambda(\sigma, d)$ induces a morphism

$$j^* \colon HH^{d+2,-d}(\Lambda(\sigma, d), \Lambda(\sigma, d)) \longrightarrow HH^{d+2}(\Lambda, {}_1\Lambda_\sigma).$$

**Theorem**

If $(\Lambda(\sigma, d), m_{d+2}, \dots)$ is a minimal model for $A$, there exists a twisted $(d+2)$-periodicity extension for $\Lambda$

$$\eta \colon \qquad {}_1\Lambda_\sigma \hookrightarrow P_{d+2} \to \cdots \to P_1 \twoheadrightarrow \Lambda$$

such that

$$j^*(\{m_{d+2}\}) = \{\eta\} \quad \text{restricted UMP (rUMP).}$$

# The key question

To which extent is $A$ determined by

- $H^*(A) = \Lambda(\sigma, d)$ and
- $\{\eta\} \in HH^{d+2}(\Lambda, {}_1\Lambda_\sigma)$?

$A$ or equivalently its minimal model

$$(\Lambda(\sigma, d), m_{d+2}, \dots, m_{di+2}, \dots).$$

## The universal Massey product

Hochschild cohomology $HH^{\bullet,*}(\Lambda(\sigma, d), \Lambda(\sigma, d))$ is a Lie algebra and a commutative algebra (Gerstenhaber algebra).

The UMP $\{m_{d+2}\} \in HH^{d+2,-d}(\Lambda(\sigma, d), \Lambda(\sigma, d))$ satisfies

$$\frac{[\{m_{d+2}\}, \{m_{d+2}\}]}{2} = 0$$

by the minimal $A_\infty$-algebra equations. We assume for simplicity that $\operatorname{char} k \neq 2$.

# The restricted universal Massey product

Since $\Lambda$ is Frobenius we have a Hochschild–Tate cohomology algebra (Eu and Schedler, 2009)

$$\underline{HH}^{\bullet,*}(\Lambda, \Lambda(\sigma, d))$$

which is defined for $\bullet < 0$ and coincides with $HH^{\bullet,*}(\Lambda, \Lambda(\sigma, d))$ for $\bullet > 0$.

The rUMP

$$\{\eta\} \in HH^{d+2}(\Lambda, {}_1\Lambda_\sigma) = \underline{HH}^{d+2,-d}(\Lambda, \Lambda(\sigma, d))$$

is a unit in $\underline{HH}^{\bullet,*}(\Lambda, \Lambda(\sigma, d))$ since the extension middle terms $P_i$ are projective-injective.

# The UMP is determined by the rUMP

**Theorem**

If $u \in HH^{d+2,-d}(\Lambda, \Lambda(\sigma, d))$ is a unit in $\underline{HH}^{\bullet,*}(\Lambda, \Lambda(\sigma, d))$ then there exists a unique $m \in HH^{d+2,-d}(\Lambda(\sigma, d), \Lambda(\sigma, d))$ such that

- $j^*(m) = u$,
- $\frac{[m,m]}{2} = 0$.

In particular, if $u = \{\eta\}$ is the rUMP then the UMP must be $\{m_{d+2}\} = m$.

# Formality

When is a DGA determined by its cohomology?

**Theorem (Kadeishvili, 1988)**

Let $B$ be a graded algebra with

$$HH^{p+2,-p}(B,B) = 0, \qquad p > 0.$$

If $A$ and $A'$ are DGAs with $H^*(A) = H^*(A') = B$ then $A$ is quasi-isomorphic to $A'$ via a quasi-isomorphism which is the identity in cohomology.

In this case there is a canonical choice for $A$, namely $B$ with trivial differential.

A $d$-sparse Massey algebra $(B, m)$ is a $d$-sparse graded algebra $B$ equipped with a Hochschild cohomology class

$$m \in HH^{d+2,-d}(B, B)$$

such that $\frac{[m,m]}{2} = 0$.

**Example**

1. $(\Lambda(\sigma, d), \{m_{d+2}\})$ and more generally
2. $(H^*(A), \{m_{d+2}\})$ where $A$ is a DGA with $d$-sparse cohomology.

The Hochschild cohomology $HH^{\bullet,*}(B,m)$ of a $d$-sparse Massey algebra $(B,m)$ is the complex with:

- cochains: $HH^{\bullet,*}(B,B)$, $\bullet \geq 2$,
- differential: $x \mapsto [m,x]$.

**Theorem**

Let $(B,M)$ be a $d$-sparse Massey algebra with

$$HH^{p+2,-p}(B,m) = 0, \qquad p > d.$$

If $A$ and $A'$ are DGAs with $d$-sparse cohomology and

$$(H^*(A),\{m_{d+2}\}) = (H^*(A'),\{m'_{d+2}\}) = (B,m)$$

then $A$ is quasi-isomorphic to $A'$ via a quasi-isomorphism which is the identity in cohomology.

**Theorem**

$$HH^{p+2,q}(\Lambda(\sigma,d),\{m_{d+2}\}) = 0, \qquad p > d, \quad q \in \mathbb{Z}.$$

**Proof.**

Multiplication by $\{m_{d+2}\}$ is a chain map on the Hochschild complex of the $d$-sparse Massey algebra $(\Lambda(\sigma,d),\{m_{d+2}\})$,

$$HH^{\bullet,*}(\Lambda(\sigma,d)) \longrightarrow HH^{\bullet+d+2,*-d}(\Lambda(\sigma,d))\colon x \mapsto \{m_{d+2}\}\cdot x.$$

It is an isomorphism for $\bullet > d+2$ since $j^*(\{m_{d+2}\})$ is a unit in $\underline{HH}^{\bullet,*}(\Lambda,\Lambda(\sigma,d))$, but it has a null-homotopy

$$HH^{\bullet,*}(\Lambda(\sigma,d)) \longrightarrow HH^{\bullet+1,*}(\Lambda(\sigma,d))\colon x \mapsto \{\delta_{/d}\}\cdot x$$

where $\delta_{/d}(x) = \frac{|x|}{d}x$ is the fractional Euler class. □

# That’s all folks!

☺Thanks for your attention!

# Bibliografía


Amiot, C. (2007). On the structure of triangulated categories with finitely many indecomposables. *Bull. Soc. Math. France*, *135*(3), 435–474. https://doi.org/10.24033/bsmf.2542

Amiot, C. (2008). *Sur les petites catégories triangulées* [Doctoral dissertation, Université Paris Diderot - Paris 7]. Paris.

Eu, C.-H., & Schedler, T. (2009). Calabi-Yau Frobenius algebras. *J. Algebra*, *321*(3), 774–815. https://doi.org/10.1016/j.jalgebra.2008.11.003

# Bibliografía


- Freyd, P. (1966). Stable homotopy. In *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)* (pp. 121–172). Springer, New York. Retrieved May 9, 2017, from http://www.ams.org/mathscinet-getitem?mr=0211399
- Hanihara, N. (2020). Auslander correspondence for triangulated categories. *Alg. Number Th.*, *14*(8), 2037–2058. https://doi.org/10.2140/ant.2020.14.2037
- Heller, A. (1968). Stable homotopy categories. *Bull. Amer. Math. Soc.*, *74*, 28–63. https://doi.org/10.1090/S0002-9904-1968-11871-3

Jasso, G., & Muro, F. (2022). The Triangulated Auslander–Iyama Correspondence. https://doi.org/10.48550/arXiv.2208.14413 with an appendix by Bernhard Keller.

Kadeishvili, T. V. (1988). The structure of the $A(\infty)$-algebra, and the hochschild and harrison cohomologies. *Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR*, *91*, 19–27. Retrieved March 22, 2021, from https://mathscinet.ams.org/mathscinet-getitem?mr=1029003

Keller, B. (1994). Deriving DG categories. *Ann. Sci. École Norm. Sup. (4)*, *27*(1), 63–102. https://doi.org/10/ghp4fw

# Bibliografía


Keller, B., & Van den Bergh, M. (2011). Deformed Calabi–Yau completions. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, *2011*(654). https://doi.org/10.1515/crelle.2011.031

Lin, Z. (2019). A general construction of n-angulated categories using periodic injective resolutions. *Journal of Pure and Applied Algebra*, *223*(7), 3129–3149. https://doi.org/10.1016/J.JPAA.2018.10.012

Muro, F. (2022). Enhanced Finite Triangulated Categories. *Journal of the Institute of Mathematics of Jussieu*, *21*(3), 741–783. https://doi.org/10.1017/S1474748020000250