The triangulated Auslander–Iyama correspondence

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Triangulated categories

Triangulated category

$\mathcal{T}$

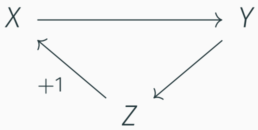
Suspension/translation

$\Sigma \colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$

Exact triangle

$X \to Y \to Z \to \Sigma X$

Exact triangle (folded)



Algebraic examples

$\mathcal{T} = D^c(A)$

$\Sigma X = X[1]$

k perfect ground field.

All categories will be additive and will have split idempotents.

Given $c \in \mathcal{C}$, add(c) $\subset \mathcal{C}$ is the full subcategory spanned by finite direct sums of direct summands of c.

$c \in \mathcal{C}$ is an additive generator if $\operatorname{add}(c) \simeq \mathcal{C}$.

$\mathcal{C}$ is finite if it has an additive generator c with $\dim \mathcal{C}(c, c) < \infty$. We can take c basic, i.e.

$$c = c_1 \oplus \cdots \oplus c_n$$

with c_i indecomposable and $c_i \not\cong c_j$ if $i \neq j$.

Algebras

Λ finite-dimensional basic Frobenius algebra.

$\mathrm{mod}(\Lambda)$ finite-dimensional (right) Λ-modules.

$\underline{\mathrm{mod}}(\Lambda)$ stable module category.

$\Omega(M)$ syzygy of a Λ-module M.

$\Lambda^e = \Lambda \otimes \Lambda^{op}$ the enveloping algebra.

Algebras

If $\sigma : \Lambda \xrightarrow{\sim} \Lambda$ is an algebra automorphism, the twisted bimodule ${}_1\Lambda_\sigma$ is Λ with the usual left action, and right action given by

$$a \cdot b = a\sigma(b).$$

Λ is twisted *n*-periodic if $\Omega^n(\Lambda) \cong {}_1\Lambda_\sigma$ in $\underline{\text{mod}}(\Lambda^e)$ for some $n > 0$ and $\sigma \in \text{Aut}(\Lambda)$.

${}_1\Lambda_\sigma \cong {}_1\Lambda_\tau$ in $\text{mod}(\Lambda^e)$ iff $[\sigma] = [\tau] \in \text{Out}(\Lambda)$.

Main theorem

Theorem (Muro, 2022)

There is a bijective correspondence between equivalence classes of:

1. Finite algebraic triangulated categories $\mathcal{T}$.
2. $(\Lambda, [\sigma])$ where:
   (a) Λ twisted 3-periodic basic Frobenius algebra.
   (b) $[\sigma] \in \mathrm{Out}(\Lambda)$ such that $\Omega^3(\Lambda) \cong {}_1\Lambda_\sigma$ in $\underline{\mathrm{mod}}(\Lambda^e)$.
3. Differential graded algebras (DGAs) A such that:
   (a) $\dim H^0(A) < \infty$.
   (b) $A \in D^c(A)$ is a basic additive generator.

The equivalence relations

1. Triangulated equivalences.
2. $(\Lambda, [\sigma]) \sim (\Lambda', [\sigma'])$ if there exists an isomorphism

$$f : \Lambda \xrightarrow{\sim} \Lambda'$$

   such that

$$[\sigma] = [f^{-1}\sigma_1 f] \in \mathrm{Out}(\Lambda).$$

3. Quasi-isomorphisms.

A related theorem

Theorem (Hanihara, 2020)

There is a bijective correspondence between equivalence classes of:

1. Finite categories $\mathcal{T}$ which can be endowed with a triangulated structure.
2. Λ twisted 3-periodic basic Frobenius algebra.

$$\mathcal{T} = D^c(A).$$

From triangulated categories to twisted periodic algebras

$\Lambda = \mathcal{T}(c,c)$ for c a basic additive generator, $\operatorname{add}(c) \simeq \mathcal{T}$. This algebra is Frobenius by Freyd, 1966.

Since $\Sigma\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ is an equivalence, $\Sigma^{-1}c \cong c$ hence the Λ-bimodule $\mathcal{T}(c,\Sigma^{-1}c)$ is twisted.

$[\sigma] \in \operatorname{Out}(\Lambda)$ is the only class such that

$$ {}_1\Lambda_\sigma \cong \mathcal{T}(c,\Sigma^{-1}c) $$

in $\operatorname{mod}(\Lambda^e)$.

The twisted periodicity isomorphism $\Omega^3(\Lambda) \cong {}_1\Lambda_\sigma$ in $\underline{\operatorname{mod}}(\Lambda^e)$ follows from Heller, 1968.

From DGAs to twisted periodic algebras

For $\mathcal{T} = D^c(A)$ with $\operatorname{add}(A) \simeq D^c(A)$ we have $\mathcal{T}(A, A) = H^0(A)$ so

$$\Lambda = H^0(A).$$

Moreover, we have $\Sigma = [1]$ and $\mathcal{T}(A, A[-1]) = H^{-1}(A)$ so

$${}_1\Lambda_\sigma \cong H^{-1}(A).$$

From twisted periodic algebras to triangulated categories

This construction is due to Amiot, 2007.

$\mathcal{T} = \mathrm{proj}(\Lambda)$.

$\Sigma^{-1} = - \otimes_\Lambda {}_1\Lambda_\sigma$ were $\Omega^3(\Lambda) \cong {}_1\Lambda_\sigma$ in $\underline{\mathrm{mod}}(\Lambda^e)$.

The previous isomorphism amounts to the existence of an exact sequence in $\mathrm{mod}(\Lambda^e)$ with projective middle terms,

$$ {}_1\Lambda_\sigma \overset{i}{\hookrightarrow} P_3 \to P_2 \to P_1 \overset{p}{\twoheadrightarrow} \Lambda. $$

From twisted periodic algebras to triangulated categories

We consider the exact sequence of Λ-bimodules

$$\begin{array}{ccccccc} & & {}_1\Lambda_\sigma & & & & \\ & \overset{p\otimes_\Lambda {}_1\Lambda_\sigma}{\nearrow\!\!\!\nearrow} & & \overset{i}{\searrow} & & & \\ P_1 \otimes_\Lambda {}_1\Lambda_\sigma & \longrightarrow & & & P_3 \longrightarrow P_2 \longrightarrow P_1 \end{array}$$

We can tensor this sequence with any $M \in \mathrm{mod}(\Lambda)$

$$\Sigma^{-1}(M \otimes_\Lambda P_1) = M \otimes_\Lambda P_1 \otimes_\Lambda {}_1\Lambda_\sigma \longrightarrow M \otimes_\Lambda P_3 \longrightarrow M \otimes_\Lambda P_2 \longrightarrow M \otimes_\Lambda P_1.$$

Exact triangles in $\mathcal{T}$ are retracts of finite direct sums of these.

Λ is connected if $\Lambda \ncong \Lambda_1 \times \Lambda_2$ con $\Lambda_i \neq 0$, $i = 1, 2$.

Proposition

If Λ is connected twisted 3-periodic and non-separable then there exists a unique $[\sigma] \in \mathrm{Out}(\Lambda)$ such that $\Omega^3(\Lambda) \cong {}_1\Lambda_\sigma$ in $\underline{\mathrm{mod}}(\Lambda^e)$

Proof.

${}_1\Lambda_\sigma \cong {}_1\Lambda_\tau$ in $\underline{\mathrm{mod}}(\Lambda^e) \Leftrightarrow {}_1\Lambda_\sigma \oplus P \cong {}_1\Lambda_\tau \oplus Q$ in $\mathrm{mod}(\Lambda^e)$ for some P, Q projective.

Since Λ is connected and non-separable, ${}_1\Lambda_\sigma$ is the only non-projective indecomposable direct summand on the left, and similarly ${}_1\Lambda_\sigma$ on the right. Hence ${}_1\Lambda_\sigma \cong {}_1\Lambda_\tau$ in $\mathrm{mod}(\Lambda^e)$, therefore $[\sigma] = [\tau]$. $\square$

☺Thanks for your attention!

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