The 1-type of Waldhausen K-theory

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(joint work with A. Tonks)

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Goal

- Understanding K_1 in the same clear way we understand K_0.

We use the following notation for the basic structure of a Waldhausen category **W**:

- Zero object $*$.
- Weak equivalences $A \xrightarrow{\sim} A'$.
- Cofiber sequences $A \rightarrowtail B \twoheadrightarrow B/A$.

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K-theory of a Waldhausen category

The *K-theory* of a Waldhausen category $\mathbf{W}$ is a spectrum $K\mathbf{W}$ and

$$K_*\mathbf{W} = \pi_* K\mathbf{W}.$$

The spectrum $K\mathbf{W}$ was defined by Waldhausen by using the $S.$-construction which associates a simplicial category $wS.\mathbf{W}$ to any Waldhausen category.

A simplicial category is regarded as a bisimplicial set by taking levelwise the nerve of a category.

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An (m, n)-bisimplex of $wS.\mathbf{W}$

$$
\begin{tikzcd}
* \arrow[r, tail] & A_{1,n} \arrow[r, tail] & A_{2,n} \arrow[r, tail] & \cdots\cdots \arrow[r, tail] & A_{m,n} \\
\vdots \arrow[u, "\sim"'] & \vdots \arrow[u, "\sim"'] & \vdots \arrow[u, "\sim"'] & \ddots & \vdots \arrow[u, "\sim"'] \\
* \arrow[r, tail] \arrow[u, "\sim"'] & A_{1,1} \arrow[r, tail] \arrow[u, "\sim"'] & A_{2,1} \arrow[r, tail] \arrow[u, "\sim"'] & \cdots\cdots \arrow[r, tail] & A_{m,1} \arrow[u, "\sim"'] \\
* \arrow[r, tail] \arrow[u, "\sim"'] & A_{1,0} \arrow[r, tail] \arrow[u, "\sim"'] & A_{2,0} \arrow[r, tail] \arrow[u, "\sim"'] & \cdots\cdots \arrow[r, tail] & A_{m,0} \arrow[u, "\sim"']
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Examples in low degrees: $m+n=1,2$ $(1,2)$ $(2,1)$ $(3,0)$

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K_0 of a Waldhausen category

The group $K_0\mathbf{W}$ is generated by the symbols

- $[A]$ for any object A in $\mathbf{W}$.

These symbols satisfy the following relations:

- $[*] = 0$,
- $[A] = [A']$ for any weak equivalence $A \xrightarrow{\sim} A'$,
- $[B/A] + [A] = [B]$ for any cofiber sequence $A \rightarrowtail B \twoheadrightarrow B/A$.

The generators and relations correspond to the bisimplices of total degree 1 and 2, respectively, in Waldhausen's $S_{\cdot}$-construction.

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How the algebraic model looks like

We are going to define a chain complex of non-abelian groups $\mathcal{D}_*\mathbf{W}$ concentrated in dimensions $n = 0, 1$ whose homology is $H_n\mathcal{D}_*\mathbf{W} \cong K_n\mathbf{W}$.

$$\begin{tikzcd}
& (\mathcal{D}_0\mathbf{W})^{ab} \otimes (\mathcal{D}_0\mathbf{W})^{ab} \arrow[d, "\langle\cdot,\cdot\rangle"'] & & \\
K_1\mathbf{W} \arrow[r, hook] & \mathcal{D}_1\mathbf{W} \arrow[r, "\partial"] & \mathcal{D}_0\mathbf{W} \arrow[r, two heads] & K_0\mathbf{W}.
\end{tikzcd}$$

The nature of our algebraic model

A stable quadratic module C consists of a diagram of groups

$$C_0^{ab} \otimes C_0^{ab} \xrightarrow{\langle -,- \rangle} C_1 \xrightarrow{\partial} C_0$$

such that

- $\langle a, b \rangle = -\langle b, a \rangle$,
- $\partial \langle a, b \rangle = -b - a + b + a$,
- $\langle \partial c, \partial d \rangle = -d - c + d + c$.

The homology groups of C are

- $H_0 C = \operatorname{Coker} \partial$,
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The algebraic model $\mathcal{D}_*\mathbf{W}$

We define $\mathcal{D}_*\mathbf{W}$ by generators and relations. This stable quadratic module is generated in dimension 0 by the symbols

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and in dimension 1 by

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The generating symbols satisfy six kinds of relations:

- The trivial relations formulas bisimplices.
- The boundary relations formulas bisimplices.
- Composition of weak equivalences formula bisimplex.
- Weak equivalences of cofiber sequences formula bisimplex.
- Composition of cofiber sequences formula bisimplex.
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A symmetric monoidal category

A stable quadratic module C gives rise to a symmetric monoidal category **smc**C with

- object set C_0,
- morphisms $(c_0, c_1)\colon c_0 \to c_0 + \partial c_1$ for $c_0 \in C_0$ and $c_1 \in C_1$.

The symmetry isomorphism is defined by the bracket

$$(c_0 + c_0', \langle c_0, c_0' \rangle)\colon c_0 + c_0' \xrightarrow{\cong} c_0' + c_0.$$

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The classifying spectrum

Segal's construction associates a classifying spectrum $B\mathbf{M}$ to any symmetric monoidal category $\mathbf{M}$.

The spectrum $B\mathbf{smc}C$ has homotopy groups concentrated in dimensions 0 and 1:

$$\begin{aligned} \pi_0 B\mathbf{smc}C &\cong H_0 C, \\ \pi_1 B\mathbf{smc}C &\cong H_1 C. \end{aligned}$$

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The main theorem

The stable quadratic module $\mathcal{D}_*\mathbf{W}$ is a model for the 1-type of $K\mathbf{W}$:

Theorem

There is a natural morphism in the stable homotopy category

$$K\mathbf{W} \longrightarrow B\mathbf{smc}\mathcal{D}_*\mathbf{W}$$

which induces isomorphisms in π_0 and π_1.

Corollary

There are natural isomorphisms

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Related work

- Deligne's Picard category of virtual objects of an exact category.
- Nenashev's presentation of K_1 of an exact category.

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Deligne's category of virtual objects

Deligne defined the category of virtual objects $V(\mathbf{E})$ of an exact category $\mathbf{E}$, which is a symmetric monoidal category with unit object $*$ such that

$$\mathrm{Iso}(V(\mathbf{E})) \cong K_0\mathbf{E}, \quad \mathrm{Aut}_{V(\mathbf{E})}(*) \cong K_1\mathbf{E}.$$

Proposition

There is an equivalence of categories

$$V(\mathbf{E}) \quad \simeq \quad \mathbf{smc}\mathcal{D}_*\mathbf{E}.$$

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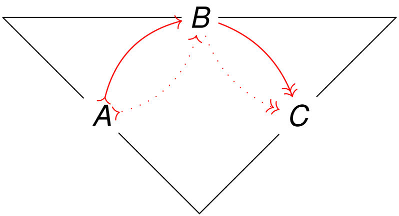
Nenashev's representatives for $K_1\mathbf{E}$

Nenashev showed that any element of $K_1\mathbf{E}$ can be represented by a pair of short exact sequences:

$$A \rightrightarrows B \rightrightarrows C.$$

The associated 2-sphere in $|wS.\mathbf{E}|$ is



A pair of weak cofiber sequences is a diagram in $\mathbf{W}$

$$\begin{tikzcd} & & C_1 & \\ A \arrow[r, shift left, tail] \arrow[r, shift right, tail] & B \arrow[ur, two heads] \arrow[dr, two heads] & & C. \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\ & & C_2 & \end{tikzcd}$$

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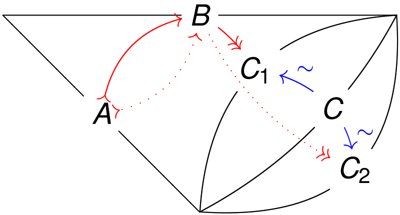

Theorem

Any element in $K_1\mathbf{W}$ is represented by a pair of weak cofiber sequences.

$$\begin{tikzcd} & & C_1 & \\ A \arrow[r, tail, shift left] \arrow[r, tail, shift right] & B \arrow[ur, two heads] \arrow[dr, two heads] & & C. \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\ & & C_2 & \end{tikzcd}$$

The element in $\mathcal{D}_1\mathbf{W}$ corresponding to the pair of weak cofiber sequences is:

$$-[C \xrightarrow{\sim} C_1] - [A \rightarrowtail B \twoheadrightarrow C_1]$$
$$+[A \rightarrowtail B \twoheadrightarrow C_2] + [C \xrightarrow{\sim} C_2] + \langle [A], -[C_2] + [C_1] \rangle.$$

Our representatives for $K_1\mathbf{W}$

Theorem

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The End

Thanks for your attention!

The trivial relations

- $[*] = 0$.
- $[A \xrightarrow{1_A} A] = 0$.
- $[A \xrightarrow{1_A} A \twoheadrightarrow *] = 0$, $[* \rightarrowtail A \xrightarrow{1_A} A] = 0$.

back

The boundary relations

- $\partial[A \xrightarrow{\sim} A'] = -[A'] + [A]$.
- $\partial[A \rightarrowtail B \twoheadrightarrow B/A] = -[B] + [B/A] + [A]$.

back

Composition of weak equivalences

- For any pair of composable weak equivalences $A \xrightarrow{\sim} A' \xrightarrow{\sim} A''$,

$$[A \xrightarrow{\sim} A''] = [A' \xrightarrow{\sim} A''] + [A \xrightarrow{\sim} A'].$$

back

- For any commutative diagram in **W** as follows

$$\begin{tikzcd} A \arrow[r, rightarrowtail] \arrow[d, "\sim"] & B \arrow[r, twoheadrightarrow] \arrow[d, "\sim"] & B/A \arrow[d, "\sim"] \\ A' \arrow[r, rightarrowtail] & B' \arrow[r, twoheadrightarrow] & B'/A' \end{tikzcd}$$

we have

$$\begin{aligned} &[A' \rightarrowtail B' \twoheadrightarrow B'/A'] \\ &[A \xrightarrow{\sim} A'] + [B/A \xrightarrow{\sim} B'/A'] \\ &+\langle [A], -[B'/A'] + [B/A] \rangle \quad = \quad [B \xrightarrow{\sim} B'] \\ &\qquad\qquad\qquad\qquad\qquad\qquad +[A \rightarrowtail B \twoheadrightarrow B/A]. \end{aligned}$$

back

Composition of cofiber sequences

- For any commutative diagram consisting of four obvious cofiber sequences in **W** as follows

$$\begin{tikzcd}
 & & C/B \\
 & B/A \arrow[r, tail] & C/A \arrow[u, two heads] \\
A \arrow[r, tail] & B \arrow[r, tail] \arrow[u, two heads] & C \arrow[u, two heads]
\end{tikzcd}$$

we have

$$\begin{aligned}
&[B \rightarrowtail C \twoheadrightarrow C/B] \\
&+[A \rightarrowtail B \twoheadrightarrow B/A] \quad = \quad [A \rightarrowtail C \twoheadrightarrow C/A] \\
&\qquad\qquad\qquad\qquad\qquad +[B/A \rightarrowtail C/A \twoheadrightarrow C/B] \\
&\qquad\qquad\qquad\qquad\qquad +\langle [A], -[C/A] + [C/B] + [B/A] \rangle.
\end{aligned}$$

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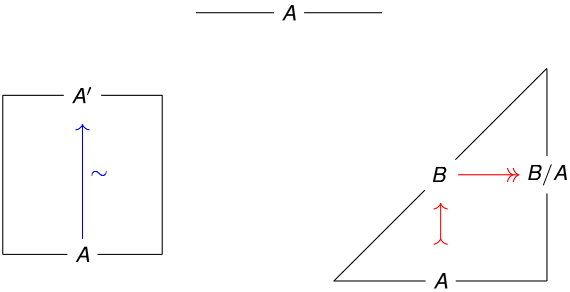
Coproducts

- For any pair of objects A, B in $\mathbf{W}$

$$\langle [A],[B]\rangle = -[A \overset{i_1}{\rightarrowtail} A\vee B \overset{p_2}{\twoheadrightarrow} B] + [B \overset{i_2}{\rightarrowtail} A\vee B \overset{p_1}{\twoheadrightarrow} A].$$

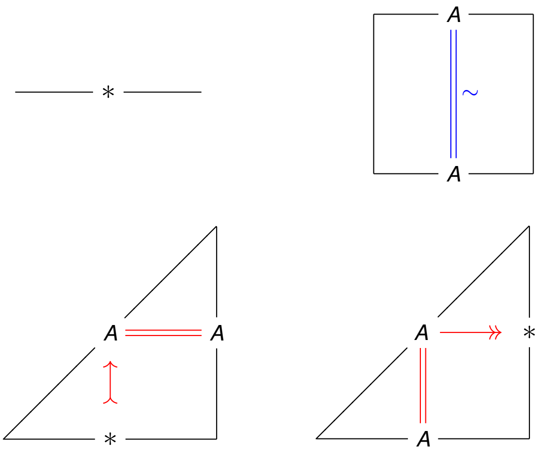
back

Bisimplices of total degree 1 and 2 in $wS.\mathbf{W}$



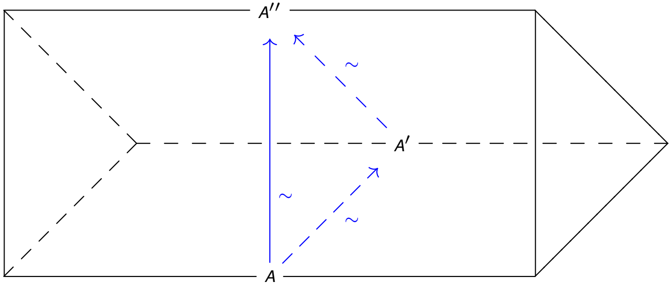
back to bisimplices | back to K_0 | back to generators | back to relations

Degenerate bisimplices of total degree 1 and 2 in $wS.\mathbf{W}$



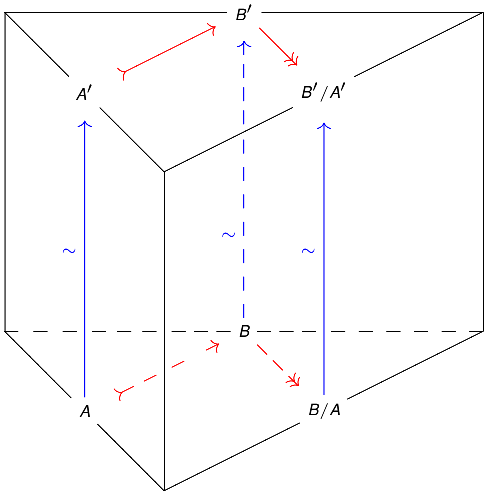
back

Bisimplex of bidegree $(1,2)$ in $wS.\mathbf{W}$



◂ back to bisimplices ◂ back to relations

Bisimplex of bidegree $(2, 1)$ in $wS.\mathbf{W}$



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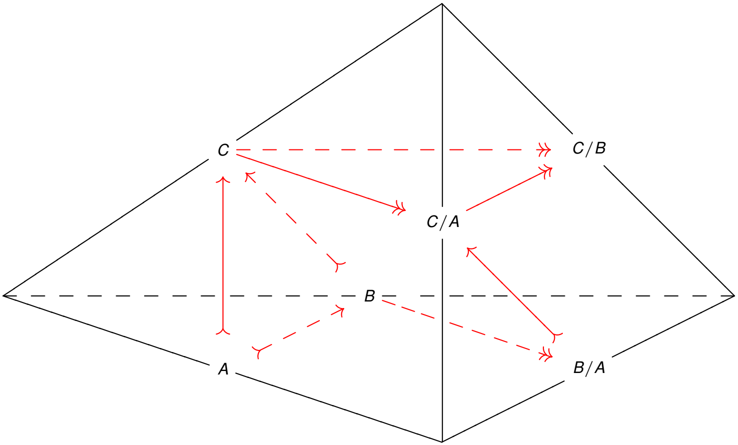
Bisimplex of bidegree $(3, 0)$ in $wS.\mathbf{W}$



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