New Directions in Group Theory and Triangulated Categories

Uniqueness of enhancements for Hom-finite triangulated categories with an n-cluster tilting object

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Assumption:

k ground field perfect

$\mathcal{T}$ additive category

small

idempotent complete

$\dim \mathcal{T}(X,Y) < \infty$

$\Sigma : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ suspension or translation functor

An algebraic triangulated structure on $\mathcal{T}$ is a DG-category $\mathcal{A}$ and an equivalence

$$\mathcal{T} \simeq D^c(\mathcal{A})$$

enhancement [Bondal-Kapranov]

A Morita equivalence between DG-categories is a DG-functor $\mathcal{A} \to \mathcal{B}$ which induces

$$D^c(\mathcal{A}) \xrightarrow{\sim} D^c(\mathcal{B}).$$

$\mathcal{T}$ is finite if it has finitely many indecomposables (up to iso.)

$\mathcal{T} \simeq \operatorname{proj}(\Lambda)$ in this case

$\Lambda = \mathcal{T}(T,T)$ $\quad$ $T = \underbrace{T_1 \oplus \cdots \oplus T_n}_{\text{indecomposable}}$

[Freyd] $\mathcal{T}$ triangulated $\Rightarrow$ Λ basic Frobenius algebra

[M. 2020]

<u>Thm</u>: $\mathcal{T} \simeq \mathrm{proj}(\Lambda)$ Λ basic Frobenius alg.

1) $\mathcal{T}$ has an enhanced triangulated structure
$\iff$ $\Omega^3_{\underline{\Lambda^e}}(\Lambda)$ stably isomorphic to an invertible Λ-bimodule I

enveloping algebra

2) If 1) holds then $\Sigma \cong - \otimes_\Lambda I^{-1}$

3) If Σ is given, any two enhancements of $(\mathcal{T}, \Sigma)$ are Morita equivalent.

[Hanihara 2020]

<u>thm</u> : $\mathcal{T} \simeq \mathrm{proj}(\Lambda)$ has an ordinary triangulated structure $\Leftrightarrow \Omega^3_{\Lambda^e}(\Lambda)$ stably isomorphic to an invertible Λ-bimodule I

Yet we do not know whether all of them are algebraic.

An n-cluster tilting object T of $(\mathcal{T}, \Sigma)$ is an object such that

$\mathcal{T}(T, \Sigma^r X) = 0 \ \forall\, 0 < r < n \iff X \in \operatorname{add}(T)$

$\mathcal{T}(\Sigma^r X, T) = 0 \quad - - - - - \iff$

T generates $\mathcal{T}$

$\mathcal{T}$ finite $\iff \exists$ 1-cluster tilting object

[J-M]

Theorem: $\mathcal{T}$ algebraic triangulated category with an n-cluster tilting object $\Rightarrow$ $\mathcal{T}$ has a unique enhancement up to Morita equivalence

[Geiss - Keller - Oppermann 2013]

An n-angulated category $\mathcal{F}$ is a category equipped with a susp. functor $\Sigma_n : \mathcal{F} \to \mathcal{F}$ on a class of n-angles

$$X_1 \to X_2 \to \cdots \to X_n \to \Sigma_n X_1$$

+ axioms
in such a way that the case $n=3$ is triangulated categories.

Thm [G-K-O] If $\mathcal{T}$ is a triangulated category and T is an $(n-2)$-cluster tilting object then

$$\mathcal{F} = \operatorname{add}(T)$$

is n-angulated with suspension Σ^{n-2}

(Σ is the susp. of $\mathcal{T}$) $\quad \in \mathcal{F}$

$$\begin{tikzcd}
 & X_2 \arrow[rr] \arrow[dr] \arrow[d, phantom, "\Delta \text{ in } \mathcal{T}"] & & X_3 \arrow[rr] \arrow[dr] \arrow[d, phantom, "\Delta \text{ in } \mathcal{T}"] & & X_4 & - - - & & X_{n-1} \arrow[dr] & \\
X_1 \arrow[ur] & & X_{2.5} \arrow[ll, "+1"'] \arrow[ur] & & X_{3.5} \arrow[ll, "+1"'] \arrow[ur] & - - - & & X_{n-1.5} \arrow[ur] & & X_n \arrow[ll, "+1"'] \arrow[lllllllll, bend left, "+(n-2)"]
\end{tikzcd}$$

[J-M] extending [Bondal-Kapranov 1991]

Def: $n \geq 3$ a DG-category $\mathcal{A}$ pre-n-angulated if the Yoneda full embedding

$$H^0(\mathcal{A}) \longrightarrow D^c(\mathcal{A})$$

is the inclusion of an $(n-2)$-cluster tilting subcategory. ← $H^0(\mathcal{A})$ is n-angulated by [G-K-O]

$H^*(\mathcal{A})$ is concentrated in degrees multiple of n

An enhanced n-angulated structure on $\mathcal{F}$ is a pre-n-angulated $\mathcal{A}$ and

$$\mathcal{F} \simeq H^0(\mathcal{A})$$

enhancement

A DG-functor $\mathcal{A} \to \mathcal{B}$ is a quasi-eq. if

$$H^*(\mathcal{A}) \xrightarrow{\sim} H^*(\mathcal{B})$$

is an equivalence of graded categories.

If $\mathrm{proj}(\Lambda)$ is n-angulated $\Rightarrow$ Λ Frobenius

Λ basic f.d. algebra

[G-K-O] extending [Freyd]

[J-M]

<u>Thm</u>: $\mathcal{F}_n \simeq \operatorname{proj}(\Lambda)$ Λ basic Frobenius alg.

1) $\mathcal{F}$ has an enhanced n-angulated structure
$\iff$ $\Omega^{n}_{\Lambda^e}(\Lambda)$ stably isomorphic to an invertible Λ-bimodule I

enveloping algebra

2) If 1) holds then $\Sigma_n \cong - \otimes_{\Lambda} I^{-1}$

3) If Σ is given, any two enhancements of $(\mathcal{F}_n, \Sigma_n)$ are quasi-equivalent.

If $\mathcal{T} = \mathcal{D}^c(\mathcal{A}) = H^0(\mathcal{A})$, $\mathcal{A}$ pre-3-angulated enhancement and $\mathcal{F} \subset \mathcal{T}$ $(n-2)$-cluster tilting subcategory then the full sub-DG-cat $\mathcal{A}_{\mathcal{F}}$ spanned by the objects of $\mathcal{F}$ is an n-angulated enhancement of $\mathcal{F}$, and $\mathcal{A}$ is the Bondal-Kapranov pre-3-angulated hull of $\mathcal{A}_{\mathcal{F}}$

<u>Corollary</u>: $\mathcal{T}$ algebraic triangulated category with an n-cluster tilting object $\Rightarrow$ $\mathcal{T}$ has a unique enhancement up to Morita equivalence