UNIVERSIDAD Ð SEVILLA 1505

Geometry of Singularities and Higher Structures

Modern Developments in Geometry and Higher Structures
Brin Mathematics Research Center, University of Maryland

June 9 – 13, 2025

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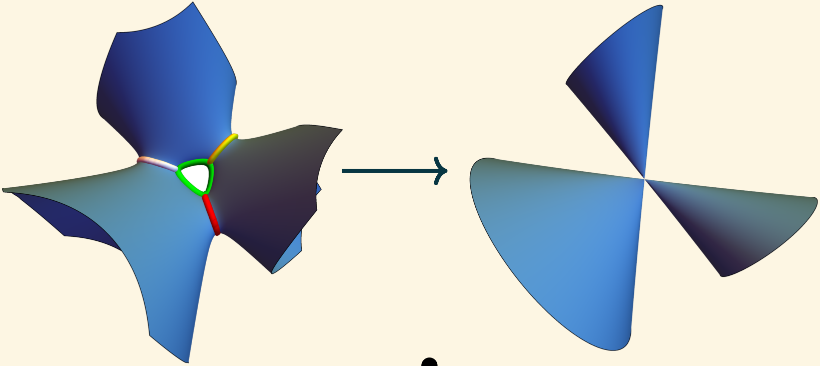
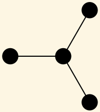
Du Val singularities[1]





D_4

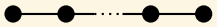
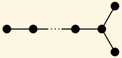
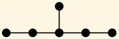
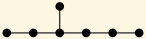
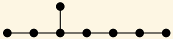
[1] Also known as Kleinian singularities or simple surface singularities.

Du Val singularities

They are isolated surface singularities arising as:

Variety		**Functions**
$\mathbb{C}^2/G$	$G \subset \mathrm{SL}(2,\mathbb{C})$ finite	$\mathbb{C}[[x,y]]^G$

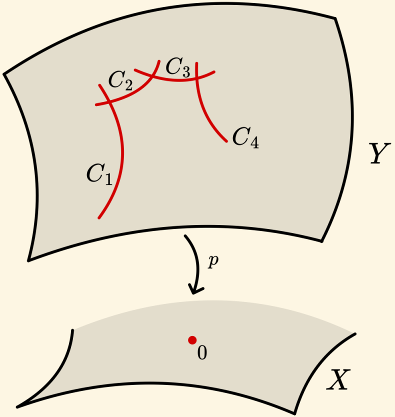
They are classified by *ADE* Dynkin diagrams:

Equation in $\mathbb{C}^3$	**Diagram**
$x^2+y^2+z^{n+1}$	 $A_n, n \geq 1$
$x^2+y^2z+z^{n-1}$	 $D_n, n \geq 4$
$x^2+y^3+z^4$	 E_6
$x^2+y^3+yz^3$	 E_7
$x^2+y^3+z^5$	 E_8

Compound Du Val singularities

A 3-dimensional
analogue of Du Val singularities:

- X complete, local, isolated hypersurface singularity at $0 \in \mathbb{C}^4$.
- The generic hyperplane section of X is a Du Val singularity.
- $p: Y \to X$ a crepant resolution, which is an isomorphism outside $p^{-1}(0) = \bigcup_{i=1}^{n} C_i$ with $C_i \cong \mathbb{P}^1$ and Y smooth.



GOAL:
Classify these cDV singularities in terms of invariants.

Deformations

A **DGA**

$\mathrm{MC}(A) = \{x \in A^1 \mid d(x) + x^2 = 0\}$ **Maurer–Cartan set**

$\mathrm{G}(A) = 1 + A^0$ **gauge group**

$\mathrm{Def}_A : \text{Artin} \to \text{Set}, \quad \mathrm{Def}_A(\Gamma) = \dfrac{\mathrm{MC}(A \otimes \mathfrak{M})}{\mathrm{G}(A \otimes \mathfrak{M})},$ **deformation functor**

$A = \mathbb{R}\mathrm{End}_{\mathrm{coh}(Y)}(\bigoplus_{i=1}^{n} \mathcal{O}_{C_i})$

Artin	Representative	
commutative	Λ_{com}	
non-commutative	Λ	***contraction algebra***
DG	Λ_{dg}	***derived contraction algebra***

$$\Lambda_{com} = \Lambda/([\Lambda,\Lambda]), \qquad \Lambda = H^0\Lambda_{dg}, \qquad \Lambda[t] = H^*\Lambda_{dg},\ |t| = -2.$$

Example (Donovan and Wemyss 2016; Booth 2019)

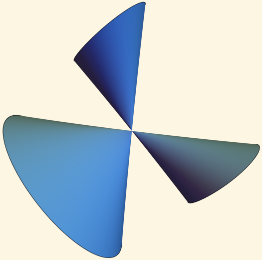
$X = \operatorname{Spec}\left(\frac{\mathbb{C}[\![x,y,u,v]\!]}{(u^2+v^2y-x^3-xy^3)}\right)$ **Laufer flop**

$\Lambda_{com} = \frac{\mathbb{C}[x,y]}{(xy\,,\,x^2-y^3)}$

$\Lambda = \frac{\mathbb{C}\langle x,y\rangle}{(xy+yx\,,\,x^2-y^3)}$ **quantum cusp**

$\Lambda_{dg} = \mathbb{C}\langle \underbrace{x, y}_{0}, \underbrace{\zeta, \xi}_{-1}, \underbrace{\rho}_{-2} \rangle,$

$d(\zeta) = xy+yx, \quad d(\xi) = x^2-y^3, \quad d(\rho) = [\xi, y]+[\zeta, x].$



Generic hyperplane section

Connection with other invariants

Theorem (Toda 2015)

If X is a cDV and $p\colon Y \to X$ contracts a single curve $p^{-1}(0) = C$ then

$$\dim_{\mathbb{C}} \Lambda = \underbrace{\dim_{\mathbb{C}} \Lambda_{com}}_{\text{Reid's width}} + \sum_{i=2}^{\text{length}} i^2 \cdot \left(i^{\text{th}} \text{ genus 0 Gopakumar–Vafa invariant} \right).$$

The Donovan–Wemyss conjecture

Conjecture (Donovan and Wemyss 2016)

Given two cDVs X, X' with crepant resolutions $Y \to X$, $Y' \to X'$ and contraction algebras Λ, Λ',

$$D(\Lambda) \simeq D(\Lambda') \iff X \cong X'.$$

Theorem (Hua and Keller 2024)

$$D_{dg}(\Lambda_{dg}) \simeq D_{dg}(\Lambda'_{dg}) \iff X \cong X'.$$

> *"Various enhanced versions of the conjecture are known to be true, but indeed the point of the conjecture is that these enhanced structures are not necessary."— Wemyss 2023*

The singularity category

Theorem (August 2020)
For X a cDV, there is a bijection between equivalence classes of:

1. Crepant resolutions $Y \to X$.
2. Derived equivalence class of contraction algebras Λ.
3. Periodic cluster-tilting objects $c \in \underbrace{D^{sg}(X) = D^b(X)/\mathrm{perf}(X)}_{\textit{singularity category}}$.

3. → 2. $\Lambda = \mathrm{End}(c)$

$\Lambda_{per} = \mathbb{R}\mathrm{End}(c)$ ***periodic contraction algebra***

$H^*\Lambda_{per} = \Lambda[t^{\pm 1}] \quad |t| = -2$

$\Lambda_{dg} = t^{\leq 0}\Lambda_{per}$ connective cover

Cluster tilting

If $\mathcal{T}$ is a small hom-finite idempotent-complete algebraic triangulated category, $c \in \mathcal{T}$ is ***periodic cluster tilting*** if

- $c \cong \Sigma^2 c$,
- $\operatorname{add}(c) = \{x \in \mathcal{T} \mid \mathcal{T}(x, \Sigma c) = 0\} = \{x \in \mathcal{T} \mid \mathcal{T}(c, \Sigma x) = 0\}$.

$\mathcal{T} \simeq \operatorname{perf}(\mathbb{R}\operatorname{End}(c)) : \; c \mapsto \mathbb{R}\operatorname{End}(c)$.

$\Lambda = \operatorname{End}(c) \cong \operatorname{End}(\Sigma^2 c)$ in two ways, induced by Σ^2 and by $c \cong \Sigma^2 c$. They compose to an automorphism $\sigma \in \operatorname{Aut}(\Lambda)$.

If X is a cDV and $\mathcal{T} = D^{sg}(X)$ then $\sigma = \operatorname{id}_\Lambda$ since $\Sigma^2 = \operatorname{id}_{\mathcal{T}}$.

The derived Auslander–Iyama correspondence

Theorem (Jasso and Muro 2023)
There is a bijection between equivalence classes of:

1. *A* ***periodic cluster-tilting DGA***, i.e.:
   a. $\operatorname{perf}(A)$ is hom-finite,
   b. $A \in \operatorname{perf}(A)$ is periodic cluster-tilting.
2. (Λ, σ) where:
   a. Λ is a basic self-injective algebra,
   b. $\sigma \in \operatorname{Out}(\Lambda)$ such that $\Omega^4_{\Lambda^e}(\Lambda) \cong \Lambda_\sigma$ in $\underline{\operatorname{mod}}(\Lambda^e)$.

1. → 2. $\Lambda = H^0 A = \operatorname{End}_{\operatorname{perf}(A)}(A)$.

$H^* A = \Lambda(\sigma)$ which is $\Lambda[t^{\pm 1}]$, $|t| = -2$, with product twisted by σ

$$t\lambda = \sigma(\lambda)t, \qquad \lambda \in \Lambda.$$

A periodic cluster-tilting DGA is formal $\Leftrightarrow \Lambda = H^0 A$ is separable.

The only cDV with separable Λ is the **Atiyah flop** $\operatorname{Spec}\left(\frac{\mathbb{C}[\![x,y,u,v]\!]}{(xy-uv)}\right)$.

Proof of the Donovan–Wemyss conjecture

Corollary (Keller in the appendix to Jasso and Muro 2023)

Given two cDVs X, X' with crepant resolutions $Y \to X$, $Y' \to X'$ and contraction algebras Λ, Λ',

$$D(\Lambda) \simeq D(\Lambda') \Longrightarrow X \cong X'.$$

Proof.

$D(\Lambda) \simeq D(\Lambda') \Rightarrow \Lambda \cong \Lambda'$	replacing one resolution (August 2020)
$\Rightarrow \Lambda_{per} \simeq \Lambda'_{per}$	by the correspondence since $\sigma = \sigma' = \mathrm{id}$
$\Rightarrow \Lambda_{dg} \simeq \Lambda'_{dg}$	taking connective covers
$\Rightarrow X \cong X'$	by Hua and Keller 2024. ∎

Universal Massey products

A DGA $\rightsquigarrow (H^*A, m_3, m_4, \dots)$ A_∞-algebra **minimal model**.

$$m_3(a_1, a_2, a_3) \in \langle [a_1], [a_2], [a_3] \rangle \quad \textbf{Masey product}.$$

The ***universal Massey product***[2] **(UMP)** of a DGA A with $H^{\mathrm{odd}}A = 0$

$$\{m_4\} \in \mathrm{HH}^{4,-2}(H^*A, H^*A), \qquad [\{m_4\}, \{m_4\}] = 0.$$

The ***restricted UMP*** **(rUMP)**

$$j^*\{m_4\} \in \mathrm{HH}^4(H^0A, H^{-2}A), \qquad j\colon\ H^0A \hookrightarrow H^*A,$$

is the first Postnikov invariant of $t^{\leq 0}A$.

A periodic cluster-tilting DGA, $j^*\{m_4\} = 0 \Leftrightarrow \Lambda = H^0A$ is separable.

[2] Baues and Dreckmann 1989; Benson, Krause, and Schwede 2004; Kaledin 2007…

Formality-like results

Theorem (Kadeishvili 1988)

If A is a DGA with $\mathrm{HH}^{n+2,-n}(H^*A, H^*A) = 0$, $n \geq 0$, then any other DGA B with $H^*B = H^*A$ is $B \simeq A$.

A **Massey algebra** (H, m) is a graded algebra H with $H^{\mathrm{odd}} = 0$ and

$$m \in \mathrm{HH}^{4,-2}(H, H), \qquad [m, m] = 0.$$

Its **Hochschild cohomology** $\mathrm{HH}^{\star,*}(H, m)$ is the cohomology of

$$(\mathrm{HH}^{\star,*}(H, H)\,,\ d = [m, -]).$$

Theorem (Jasso and Muro 2023)

If A is a DGA with $H^{\mathrm{odd}}A = 0$ and $\mathrm{HH}^{n+2,-n}(H^*A, \{m_4\}) = 0$, $n \geq 3$, then any other DGA B with $(H^*B, \{m_4^B\}) = (H^*A, \{m_4^A\})$ is $B \simeq A$.

The Bousfield–Kan spectral sequence

Both theorems are proved by using the spectral sequence associated to the mapping space in the category of operads

$$\operatorname{Map}(\mathcal{A}_\infty, \mathcal{E}(H^*A)) = \lim_n \operatorname{Map}(\mathcal{A}_n, \mathcal{E}(H^*A))$$

of A_∞-algebra structures on H^*A, pointed at the minimal model,

$$E_2^{pq} = \mathrm{HH}^{p+2,-q}(H^*A, H^*A) \Rightarrow \pi_{q-p} \operatorname{Map}(\mathcal{A}_\infty, \mathcal{E}(H^*A)).$$

If $H^{\mathrm{odd}}A = 0$ then $d_2 = 0$ and $d_3 = [\{m_4\}, -]$ so

$$E_4^{pq} = \mathrm{HH}^{p+2,-q}(H^*A, \{m_4\}).$$

Characterization of periodic cluster-tilting DGAs

Proposition (Jasso and Muro 2023)

TFAE:

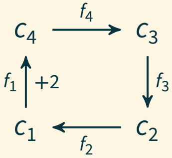
1. A is a periodic cluster-tilting DGA.
2. The following properties hold:
   a. H^0A is a finite-dimensional basic self-injective algebra,
   b. $H^{\text{odd}}A = 0$,
   c. $H^*A \cong H^{*+2}A$ as H^*A-bimodules.
   d. $j^*\{m_4\} \in \underbrace{\widehat{\mathrm{HH}}^{\star}(H^0A, H^*A)}_{\textit{Hochschild–Tate}}$ is a unit.

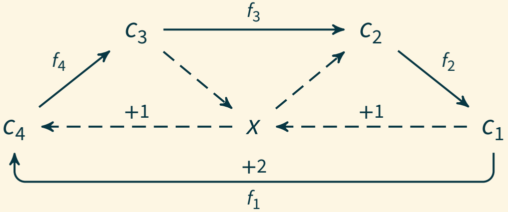
4-angulated cats. (Geiss, Keller, and Oppermann 2013)

Why is the rUMP a Hochschild–Tate unit?

Given $c \in \mathcal{T}$ periodic cluster tilting, $\mathrm{add}(c) \subset \mathcal{T}$ is a ***4-angulated category*** with shift Σ^2 and

$$c_4 \xrightarrow{f_4} c_3 \xrightarrow{f_3} c_2 \xrightarrow{f_2} c_1 \xrightarrow{f_1} \Sigma^2 c_4, \quad \langle f_1, f_2, f_3, f_4 \rangle \ni 1, \quad \textbf{\textit{4-angles}},$$





Hochschild cohomology computations

Injectivity of Auslander–Iyama correspondence from gerenalized Kadeishvili thm.

All periodic cluster-tilting DGAs with given cohomology have essentially the same UMP since

$$\{m \in \mathrm{HH}^{4,-2}(H^*A, H^*A) \mid [m,m] = 0,\ j^*m \in \widehat{\mathrm{HH}}^{\bigstar}(H^0A, H^*A)^{\times}\}$$

is an $\mathrm{Aut}(H^*A)$-orbit.

$$\mathrm{HH}^{\bigstar,*}(H^\bullet A, H^\bullet A) = \mathrm{HH}^{\bigstar}(H^0A, H^*A)^{\langle\sigma\rangle}[\delta], \quad \delta \textbf{ Euler}, \quad [\delta, x] = \frac{|x|_*}{2}x.$$

In $(\mathrm{HH}^{\bigstar,*}(H^\bullet A, H^\bullet A)\,,\ d = [\{m_4\}, -])$ we set

$$f(x) = \{m_4\} \cdot x, \qquad h(x) = \delta \cdot x, \qquad f = dh + hd,$$

$$\{m_4\} \cdot x = [\{m_4\}, \delta \cdot x] + \delta \cdot [\{m_4\}, x] \quad \textbf{\textit{Gerstenhaber relation}}.$$

The map f is an isomorphism in $\bigstar \geq 2$, so $\mathrm{HH}^{\geq 5,*}(H^\bullet A, \{m_4\}) = 0$.

THE END

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