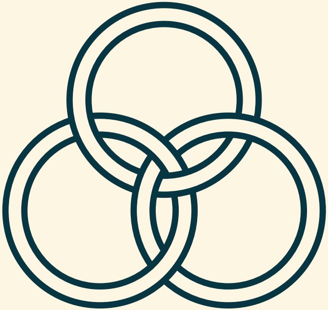

Massey products and higher operations

Operads in Málaga, 23–26 January 2023

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Classical Massey products

Given a differential graded ***associative*** algebra A and

$$a, b, c \in H^*(A), \qquad ab = 0, \qquad bc = 0,$$

their ***Massey product*** is

$$\langle a, b, c \rangle \subset H^{|a|+|b|+|c|-1}(A).$$

If $a = [\alpha]$, $b = [\beta]$, $c = [\gamma]$, choose trivializing cochains

$$d(\zeta) = \alpha\beta, \qquad d(\xi) = \beta\gamma,$$

$$\left[\zeta\gamma - (-1)^{|\alpha|}\alpha\xi\right] \in \langle a, b, c \rangle,$$

$$d\left(\zeta\gamma - (-1)^{|\alpha|}\alpha\xi\right) = (\alpha\beta)\gamma - \alpha(\beta\gamma) = 0$$

by ***associativity***.

Classical Massey products

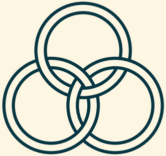
The Massey product is a coset

$$\langle a, b, c \rangle \in \frac{H^{|a|+|b|+|c|-1}(A)}{H^{|a|+|b|-1}(A) \cdot c + a \cdot H^{|b|+|c|-1}(A)}.$$

The denominator is called ***indeterminacy***.

Example

If M is the complement of the Borromean link, $H^1(M)$ is generated by three classes a, b, c such that $\langle a, b, c \rangle$ is defined, fully determined, and nontrivial.



Massey products and minimal models

Given a minimal A_∞-model of A

$$(H^*(A), m_3, m_4, \dots, m_n, \dots),$$

$$m_n \colon H^*(A) \otimes \overset{n}{\cdots} \otimes H^*(A) \longrightarrow H^*(A), \quad |m_n| = 2 - n,$$

it is well known that

$$m_3(a, b, c) \in \langle a, b, c \rangle$$

whenever the Massey product is defined.

Therefore m_3 is a replacement of Massey products with the following advantages:

- Always defined.
- No indeterminacy.

Massey products and Hochschild cohomology

In a minimal A_∞-model of A

$$(H^*(A), m_3, m_4, \dots, m_n, \dots),$$

the operation m_3 is a Hochschild cocycle. Its cohomology class, studied by Benson, Krause, and Schwede 2004,

$$[m_3] \in HH^{3,-1}(H^*(A))$$

is called ***universal Massey product*** since, for any other representative

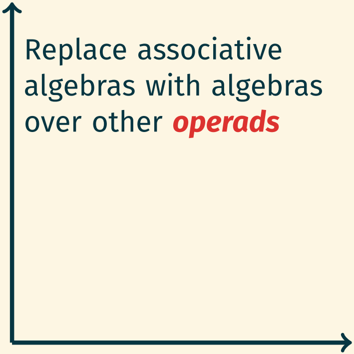
$$[\phi] = [m_3], \qquad \phi\colon H^*(A) \otimes H^*(A) \otimes H^*(A) \to H^*(A),$$

we also have

$$\phi(a, b, c) \in \langle a, b, c \rangle$$

whenever the Massey product is defined.

Beyond classical Massey products

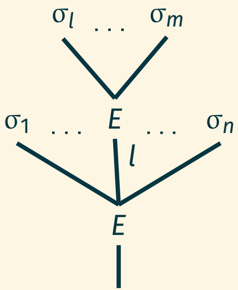


Replace associative algebras with algebras over other ***operads***

Consider ***longer*** Massey products $\langle x_1, x_2, \ldots, x_n \rangle$

Quadratic operads

Let $\mathcal{O} = (E, R)$ be a ***quadratic Koszul graded operad*** with generating reduced Σ-module $E = \{E(n)\}_{n \geqslant 0}$ and relations sub-Σ-module R

$R \subset E \circ_{(1)} E = \bigoplus_{T}$ 

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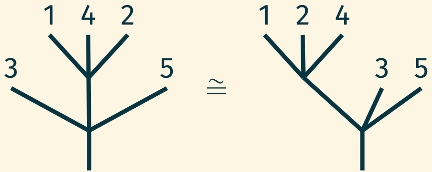
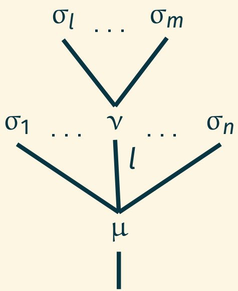


Quadratic operads

Hence a relation $\Gamma \in R(n)$ looks like

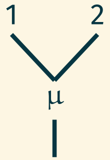
$$\Gamma = \sum$$



with $\mu, \nu \in E$.

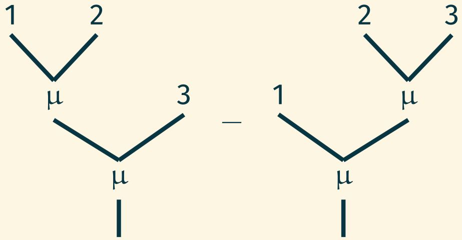
Associative operad

Generator (***product***)



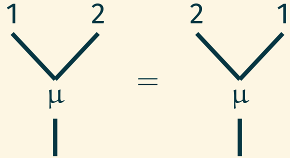
$|\mu| = 0$

Relation (***associativity***)



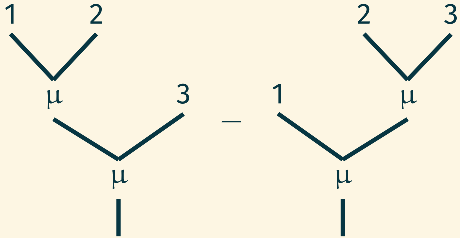
Commutative operad

Generator (***commutative product***)



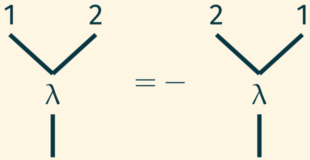
$|\mu| = 0$

Relation (***associativity***)



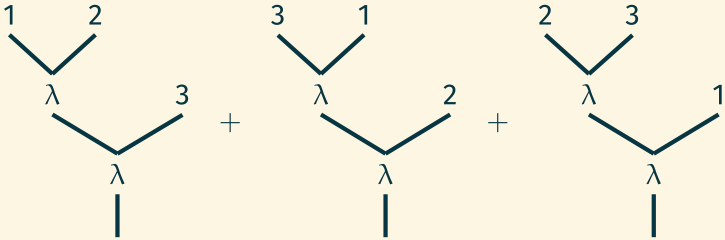
Lie operad

Generator (***Lie bracket***)



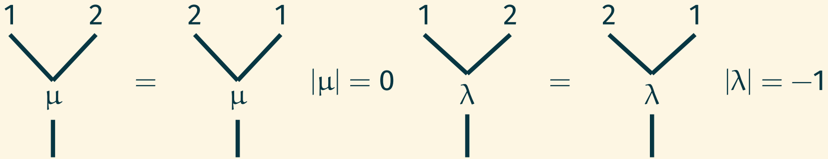
$|\lambda| = 0$

Relation (***Jacobi identity***)

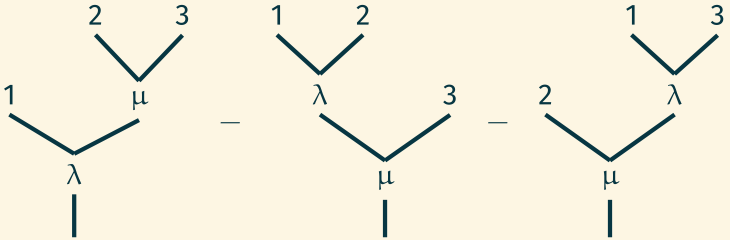


Gerstenhaber operad

Generators: commutative product and ***shifted Lie bracket***,



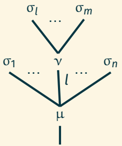
Relations: associativity, Jacobi identity, and ***Gerstenhaber relation***



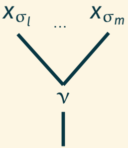
Operadic Massey products

Given a differential graded $\mathcal{O}$-algebra A, a relation

$$\Gamma = \sum$$



and elements $x_1, \dots, x_n \in H^*(A)$ such that



$$= 0$$

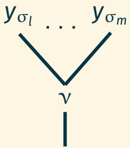
for all terms in Γ, we have an ***operadic Massey product***

$$\langle x_1, \dots, x_n \rangle_\Gamma \subset H^{\sum_{i=1}^n |x_i| + |\Gamma| - 1}(A).$$

Operadic Massey products

If $x_i = [y_i]$, choose trivializing cochains

$d(\rho) =$



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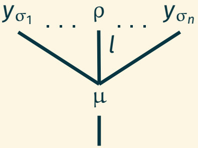
Then

$\Bigg[\sum \pm$



$\Bigg] \in \langle x_1, \ldots, x_n \rangle_\Gamma.$

Operadic Massey products

Example

- $\mathcal{O} =$ associative operad and $\Gamma =$ associativity relation recovers classical Massey products.
- $\mathcal{O} =$ Lie operad and $\Gamma =$ Jacobi identity recovers the Lie-Massey products of Retah 1977.
- Let $M = G/H$ be the Heisenberg manifold, i.e. the quotient of the Heisenberg group G of matrices

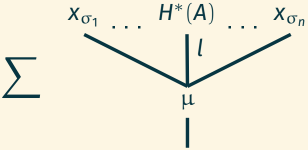
$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad a, b, c \in \mathbb{R},$$

  by the subgroup H with $a, b, c \in \mathbb{Z}$. An invariant Poisson structure on M makes $\Omega^*(M)$ a Gerstenhaber algebra with a non-trivial Massey product in $H^*(M, \mathbb{R})$ associated to the Gerstenhaber relation.

Operadic Massey products

Proposition

Given a differential graded $\mathcal{O}$-algebra A, the Massey product $\langle x_1, \ldots, x_n \rangle_\Gamma \subset H^*(A)$ is a coset with indeterminacy



Operadic Massey products and minimal models

A differential graded $\mathcal{O}$-algebra A has a minimal model, which is an $\mathcal{O}_\infty$-algebra structure on $H^*(A)$. Such a structure consists of degree $+1$ maps

$$\begin{aligned} \mathcal{O}^{\text{i}}(n) \otimes H^*(A) \otimes \overset{n}{\cdots} \otimes H^*(A) &\longrightarrow H^*(A) \\ \phi \otimes x_1 \otimes \cdots \otimes x_n &\mapsto \phi(x_1, \dots, x_n), \end{aligned}$$

where $\mathcal{O}^{\text{i}}$ is the Koszul dual of $\mathcal{O}$, which satisfies

$$(\mathcal{O}^{\text{i}})^{(1)} = E[1], \qquad (\mathcal{O}^{\text{i}})^{(2)} = R[2].$$

Theorem

Given $x_1, \dots, x_n \in H^*(A)$,

$$\Gamma[2](x_1, \dots, x_n) \in \langle x_1, \dots, x_n \rangle_\Gamma$$

whenever the operadic Massey product is defined.

Operadic Massey products and cohomology

Given an $\mathcal{O}$-algebra B, the ***operadic cohomology***

$$H^{\bullet,*}_{\mathcal{O}}(B)$$

is computed from a complex

$$C^{s,t}_{\mathcal{O}}(B) = \prod_{n\geqslant 0} \mathrm{Hom}^{s+t}((\mathcal{O}^{¡})^{(s)}(n) \otimes_{\Sigma_n} B^{\otimes^n}, B)$$

with bidegree $(+1, 0)$ differential.

The minimal model of a differential graded $\mathcal{O}$-algebra A defines a ***universal operadic Massey product*** (Dimitrova 2012)

$$[(m_{2,n})_{n\geqslant 0}] \in H^{2,-1}_{\mathcal{O}}(H^*(A))$$

represented by the minimal model operations

$$m_{2,n}\colon (\mathcal{O}^{¡})^{(2)}(n) \otimes_{\Sigma_n} H^*(A)^{\otimes^n} \longrightarrow H^*(A), \qquad n \geqslant 0.$$

Operadic Massey products and cohomology

Theorem

Let A be a differential graded $\mathcal{O}$-algebra and $x_1, \ldots, x_n \in H^*(A)$. For any other representative of the universal operadic Massey product

$$[(\phi_n)_{n\geqslant 0}] \in H^{2,-1}_{\mathcal{O}}(H^*(A)), \qquad \phi_n \colon R(n)[2] \otimes_{\Sigma_n} H^*(A)^{\otimes^n} \to H^*(A),$$

and any relation $\Gamma \in R(n)$ we also have

$$\phi_n(\Gamma[2] \otimes x_1 \otimes \cdots \otimes x_n) \in \langle x_1, \ldots, x_n \rangle_\Gamma$$

whenever the operadic Massey product is defined.

Long Massey products

If A is a differential graded associative algebra and $x_1, \dots, x_n \in H^*(A)$, the ***Massey product of length** n* is

$$\langle x_1, \dots, x_n \rangle \subset H^{\sum_{i=1}^{n} |x_i| + 2 - n}(A).$$

- $\langle x_1, x_2 \rangle = \{\pm x_1 x_2\}$.
- $\langle x_1, x_2, x_3 \rangle$ is the classical Massey product.
- $\langle x_1, \dots, x_n \rangle$ may be empty. It is non-empty iff

$$0 \in \langle x_i, x_{i+1}, \dots, x_{i+j} \rangle, \qquad j < n - 1.$$

- The indeterminacy is unknown in general.

Long Massey products and minimal models

Given a minimal A_∞-model of A

$$(H^*(A), m_3, m_4, \dots, m_n, \dots),$$

$$m_n \colon H^*(A) \otimes \overset{n}{\cdots} \otimes H^*(A) \longrightarrow H^*(A), \quad |m_n| = 2 - n,$$

it was long believed that

$$\pm m_n(x_1, \dots, x_n) \in \langle x_1, \dots, x_n \rangle$$

whenever the Massey product of length n is defined.

Example (Buijs, Moreno-Fernández, and Murillo 2020)

The previous ‘equation’ does not hold in the Sullivan model of a space $S^5 \times S^5 \times Y$, where Y fits in a fibration

$$S^5 \times S^5 \times S^5 \times S^7 \times S^7 \to Y \to S^3 \times S^3 \times S^3 \times S^3.$$

Long Massey products and minimal models

Theorem (Buijs, Moreno-Fernández, and Murillo 2020)

Given $x_1, \ldots, x_n \in H^*(A)$, if $m_i = 0$ for $2 \leqslant i \leqslant n-2$ then

$$\pm m_n(x_1, \ldots, x_n) \in \langle x_1, \ldots, x_n \rangle$$

whenever the Massey product of length n is defined.

If $H^*(A)$ is unital then $m_2 \neq 0$. Nevertheless, this is not a limitation if A is augmented.

Sparse cohomology

Assume $H^*(A)$ is concentrated in degrees $d\mathbb{Z}$. For degree reasons,

$$m_n = 0, \qquad d \nmid 2 - n.$$

Hence a minimal A_∞-model of A looks like

$$(H^*(A), m_{d+2}, m_{2d+2}, \ldots, m_{id+2}, \ldots).$$

Theorem (Jasso and Muro 2022)

In the previous situation, given $x_1, \ldots, x_{d+2} \in H^*(A)$

$$\pm m_{d+2}(x_1, \ldots, x_{d+2}) \in \langle x_1, \ldots, x_{d+2} \rangle$$

whenever the Massey product of length $d+2$ is defined.

Sparse cohomology

If $H^*(A)$ is concentrated in degrees $d\mathbb{Z}$, in a minimal A_∞-model of A

$$(H^*(A), m_{d+2}, m_{2d+2}, \dots, m_{id+2}, \dots)$$

the operation m_{d+2} is a Hochschild cocycle. Its cohomology class

$$[m_{d+2}] \in HH^{d+2,-d}(H^*(A))$$

is called ***universal Massey product of length*** $d+2$.

Proposition (Jasso and Muro 2022)

For any other representative

$$[\phi] = [m_{d+2}], \qquad \phi\colon H^*(A)^{\otimes^{d+2}} \to H^*(A),$$

if the Massey product of length $d+2$ is defined then

$$\phi(x_1, \dots, x_{d+2}) \in \langle x_1, \dots, x_{d+2} \rangle.$$

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