HOW MANY UNITS MAY AN A_∞-ALGEBRA HAVE?

Workshop in Category Theory and Algebraic Topology, Louvain-la-Neuve, 10–12 September 2015.

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Overview

1. The space of units
2. How to compute it
3. Specific cases
   3.1 Groupoids
   3.2 Chain complexes
4. Transfer to other categories

The space of units

The space of units

Being unital is a property rather than a structure.

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$$\mathtt{uAs}_{\mathcal{V}} \xrightarrow{\text{forget } 1} \mathtt{As}_{\mathcal{V}}$$

Closed symmetric monoidal category

The space of units

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$$\mathtt{uAs}_{\mathcal{V}} \xrightarrow{\text{forget } 1} \mathtt{As}_{\mathcal{V}}$$

Unital associative algebras in $\mathcal{V}$

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Associative algebras in $\mathcal{V}$

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Faithful

The space of units

Being unital is a property rather than a structure.

$$\texttt{uAs}_{\mathcal{V}}^{\text{iso}} \xrightarrow{\text{forget } 1} \texttt{As}_{\mathcal{V}}^{\text{iso}}$$

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Fully faithful!

The space of units

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Closed symmetric monoidal model category

The space of units

Being unital is a property rather than a structure.

$$\mathtt{uAs}_{\mathcal{V}}^{\mathrm{we}} \xrightarrow{\text{forget } 1} \mathtt{As}_{\mathcal{V}}^{\mathrm{we}}$$

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$$|\mathtt{uAs}_{\mathcal{V}}^{\mathrm{we}}| \xrightarrow{\text{forget } 1} |\mathtt{As}_{\mathcal{V}}^{\mathrm{we}}|$$

The space of units

Being unital is a property rather than a structure.

$$|\mathtt{uAs}^{\mathrm{we}}_{\mathcal{V}}| \xrightarrow{\text{forget } 1} |\mathtt{As}^{\mathrm{we}}_{\mathcal{V}}|$$

A vertex is an associative algebra A

The space of units

Being unital is a property rather than a structure.

$$\boxed{\text{Space of units of } A} \xrightarrow{\text{fiber at } A} |\mathtt{uAs}^{\text{we}}_{\mathcal{V}}| \xrightarrow{\text{forget } 1} |\mathtt{As}^{\text{we}}_{\mathcal{V}}|$$

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$$\text{Space of units of } A \xrightarrow{\text{fiber at } A} |\mathtt{uAs}_{\mathcal{V}}^{\text{we}}| \xrightarrow{\text{forget } 1} |\mathtt{As}_{\mathcal{V}}^{\text{we}}|$$

Theorem

The space of units is either empty or contractible if $\mathcal{V}$ is simplicial, complicial, or spectral.

How to compute it

The space of units

$$\begin{array}{ccc} \mathtt{uAs}_{\mathcal{V}} & \longrightarrow & \mathcal{V} \\ \big\downarrow & \text{forgetful functors} & \big\| \\ \mathtt{As}_{\mathcal{V}} & \longrightarrow & \mathcal{V} \end{array}$$

The space of units

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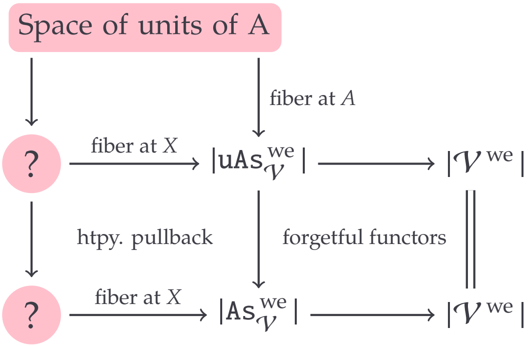
\end{array}
$$

The space of units

$$\begin{array}{ccccc} ? & \xrightarrow{\text{fiber at } X} & |\mathtt{uAs}_{\mathcal{V}}^{\text{we}}| & \longrightarrow & |\mathcal{V}^{\text{we}}| \\ \downarrow & \text{htpy. pullback} & \downarrow & \text{forgetful functors} & \| \\ ? & \xrightarrow{\text{fiber at } X} & |\mathtt{As}_{\mathcal{V}}^{\text{we}}| & \longrightarrow & |\mathcal{V}^{\text{we}}| \end{array}$$

Here A is an associative algebra with underlying object X.

The space of units



Here A is an associative algebra with underlying object X.

The space of algebra structures

$$\mathtt{As}^{\mathrm{iso}}_{\mathcal{V}} \xrightarrow{\text{forget}} \mathcal{V}^{\mathrm{iso}}$$

Faithful

The space of algebra structures

$$|\mathtt{As}_{\mathcal{V}}^{\mathrm{we}}| \xrightarrow{\text{forget}} |\mathcal{V}^{\mathrm{we}}|$$

The space of algebra structures

$$\boxed{\text{Space of associative algebra structures on } X} \xrightarrow{\text{fiber at } X} |\mathtt{As}_{\mathcal{V}}^{\mathrm{we}}| \xrightarrow{\text{forget}} |\mathcal{V}^{\mathrm{we}}|$$

The space of algebra structures

$$\text{Space of associative algebra structures on } X \xrightarrow{\text{fiber at } X} |\mathtt{As}_{\mathcal{V}}^{\mathrm{we}}| \xrightarrow{\text{forget}} |\mathcal{V}^{\mathrm{we}}|$$

Theorem (Rezk'96, M'15)

The space of associative algebra structures on a fibrant and cofibrant object X is

$$\mathrm{Map}_{\mathrm{Op}}(\mathtt{As}, \mathtt{End}(X)).$$

Op = the category of nonsymmetric operads in $\mathcal{V}$.

$\mathtt{As}$ = the associative operad in $\mathcal{V}$.

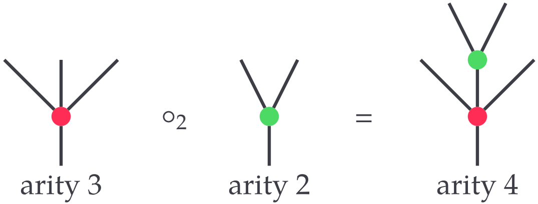
$\mathtt{End}(X)$ = the endomorphism operad of X.

Definition

An OPERAD $\mathrm{P} = \{\mathrm{P}(n)\}_{n\geq 0}$ is a sequence of objects in $\mathcal{V}$ equipped with composition operations

$$\circ_i : \mathrm{P}(s) \otimes \mathrm{P}(t) \longrightarrow \mathrm{P}(s+t-1), \qquad 1 \leq i \leq s,$$

and an identity in arity 1 satisfying the laws of tree grafting.



Operads

Definition

An operad $\mathrm{P} = \{\mathrm{P}(n)\}_{n \geq 0}$ is a sequence of objects in $\mathcal{V}$ equipped with composition operations

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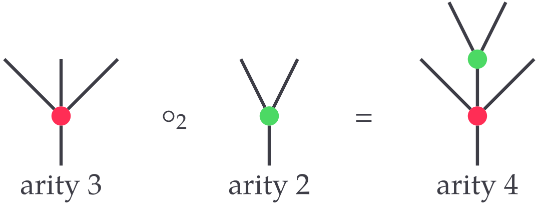

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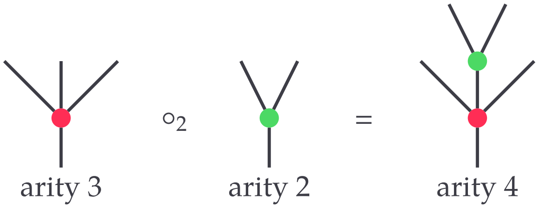

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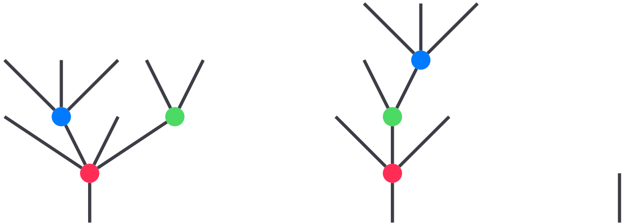


associativity

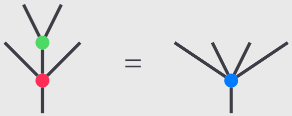
identity

Operads

Example

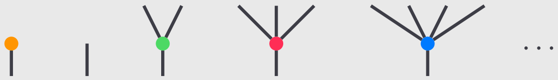
The ASSOCIATIVE OPERAD $\mathtt{As}$ consists of

Composition away from the identity is given by grafting and contracting



Example

The UNITAL ASSOCIATIVE OPERAD uAs consists of



Composition away from the identity is given by grafting and contracting, except for

Example

The ENDOMORPHISM OPERAD of an object X in $\mathcal{V}$,

$$\mathtt{End}(X)(n) = \mathtt{Hom}_{\mathcal{V}}(X^{\otimes n}, X).$$

Operads

Example

The ENDOMORPHISM OPERAD of an object X in $\mathcal{V}$,

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Definition

A $\mathtt{P}$-ALGEBRA is a map of operads $\mathtt{P} \to \mathtt{End}(X)$.

Operads

Example

The ENDOMORPHISM OPERAD of an object X in $\mathcal{V}$,

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Definition

A P-ALGEBRA is a map of operads $\mathtt{P} \to \mathtt{End}(X)$.

Theorem (Rezk, Hinich, Berger–Moerdijk…Lyubashenko, M.)

The category Op of operads in $\mathcal{V}$ inherits a model structure.

The space of algebra structures

$$\boxed{\text{Space of P-algebra structures on } X} \xrightarrow{\text{fiber at } X} |\mathtt{P}_{\mathcal{V}}^{\mathrm{we}}| \xrightarrow{\text{forget}} |\mathcal{V}^{\mathrm{we}}|$$

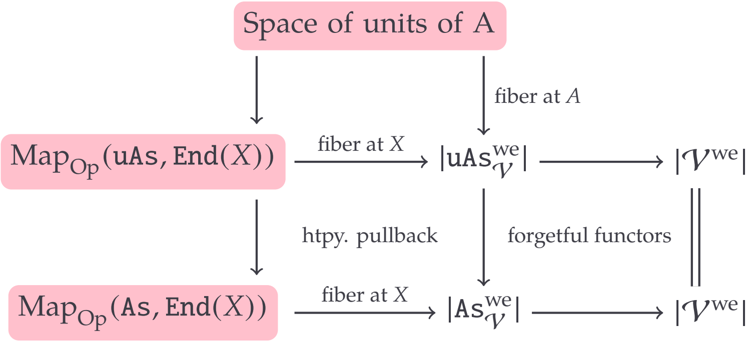
Theorem (Rezk'96, M'15)

The space of $\mathtt{P}$-algebra structures on a fibrant and cofibrant object X is

$$\mathrm{Map}_{\mathrm{Op}}(\mathtt{P}, \mathtt{End}(X)).$$

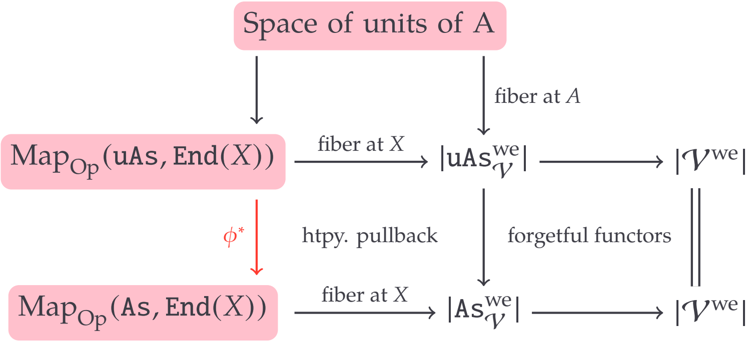
$\mathtt{P}$ = a nonsymmetric operad in $\mathcal{V}$ with cofibrant components, e.g. $\mathtt{P} = \mathtt{As}$ or $\mathtt{uAs}$, the unital associative operad.

The spaces of units and algebra structures



Here A is an associative algebra with underlying object X.

The spaces of units and algebra structures



Here A is an associative algebra with underlying object X.

The RED MAP is induced by the canonical map

$$\phi\colon \mathtt{As} \longrightarrow \mathtt{uAs}.$$

A homotopy epimorphism

Theorem

For any operad $\mathtt{P}$ in $\mathcal{V}$, the fibers of the following map are either empty or contractible,

$$\phi^*\colon \operatorname{Map}_{\mathrm{Op}}(\mathtt{uAs},\mathtt{P}) \longrightarrow \operatorname{Map}_{\mathrm{Op}}(\mathtt{As},\mathtt{P}).$$

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A map $f\colon X \to Y$ in a category C is an EPIMORPHISM if any of these equivalent statements holds:

- $f^*\colon \mathtt{Hom}_C(Y, Z) \to \mathtt{Hom}_C(X, Z)$ is injective for any Z in C.
- In the following pushout the RED ARROWS are isomorphisms

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f}\downarrow & \text{pushout} & \downarrow \\ Y & \longrightarrow & Y \bigcup_X Y \end{array}$$

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A map $f\colon X \to Y$ in a MODEL category C is a HOMOTOPY EPIMORPHISM if any of these equivalent statements holds:

- $f^*\colon \operatorname{Map}_C(Y,Z) \to \operatorname{Map}_C(X,Z)$ has empty or contactible fibers for any Z in C.
- In the following homotopy pushout the RED ARROWS are w.e.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f}\downarrow & \text{htpy. push.} & \downarrow \\ Y & \longrightarrow & Y \bigcup_X^{\mathbb{L}} Y \end{array}$$

A homotopy epimorphism

Theorem

For any operad $\mathtt{P}$ in $\mathcal{V}$, the fibers of the following map are either empty or contractible,

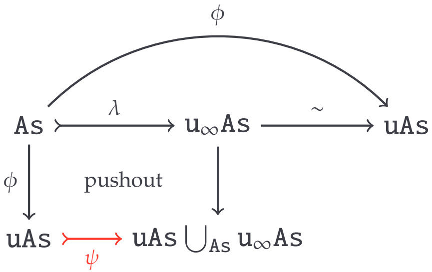
$$\phi^*\colon \operatorname{Map}_{\mathrm{Op}}(\mathtt{uAs},\mathtt{P}) \longrightarrow \operatorname{Map}_{\mathrm{Op}}(\mathtt{As},\mathtt{P}).$$

Equivalently, one (and hence both) of the two RED MAPS in the following homotopy pushout in Op is a weak equivalence,

$$\begin{array}{ccc}
\mathtt{As} & \xrightarrow{\ \phi\ } & \mathtt{uAs} \\
{\scriptstyle\phi}\downarrow & \text{htpy. pushout} & \downarrow \\
\mathtt{uAs} & \longrightarrow & \mathtt{uAs} \bigcup^{\mathbb{L}}_{\mathtt{As}} \mathtt{uAs}
\end{array}$$

The homotopy pushout

If Op were LEFT PROPER, the previous homotopy pushout would be the following pushout



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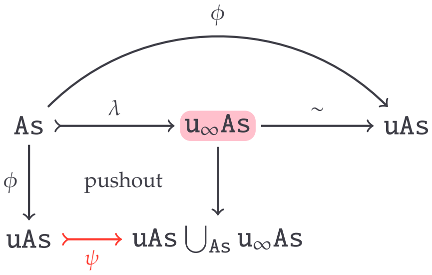


The u_∞ ASSOCIATIVE OPERAD.

Relative left properness

Theorem

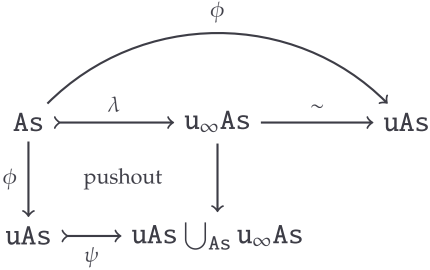
Given a pushout diagram in Op

$$\begin{array}{ccc} \mathsf{P} & \rightarrowtail & \mathsf{R} \\ {\scriptstyle f}\downarrow & \text{pushout} & \downarrow{\scriptstyle g} \\ \mathsf{Q} & \rightarrowtail & \mathsf{Q} \bigcup_{\mathsf{P}} \mathsf{R} \end{array}$$

such that the components of P and Q are cofibrant in $\mathcal{V}$, if f is a weak equivalence then so is g.

Specific cases

What are algebras over these operads?



In the category of groupoids

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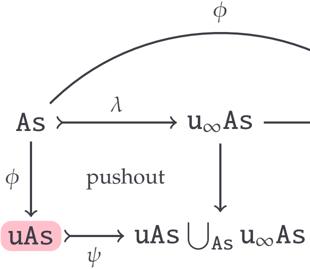
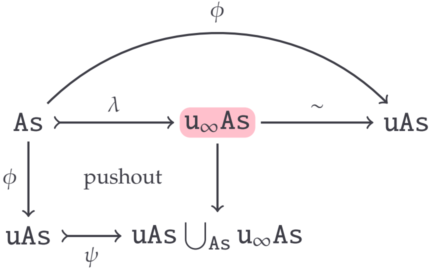


Strictly associative and strictly unital monoidal groupoids

In the category of groupoids

What are algebras over these operads?



Strictly associative monoidal groupoids

In the category of groupoids

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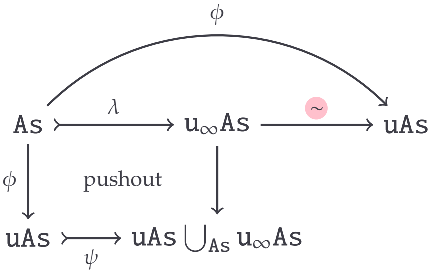
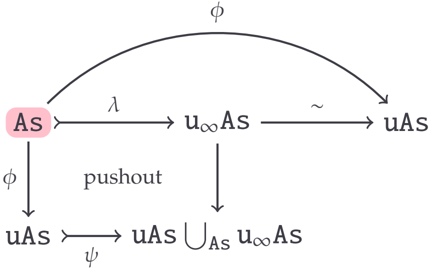


The tensor unit can be canonically strictified

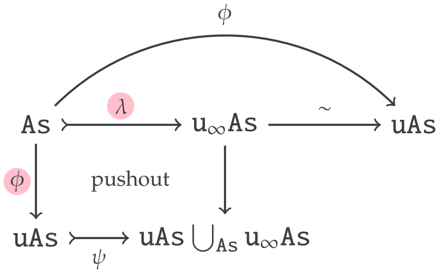
In the category of groupoids

What are algebras over these operads?



Strictly associative non-unital monoidal groupoids

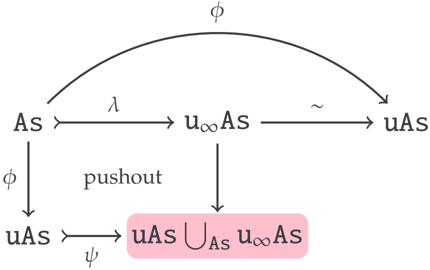
What are algebras over these operads?



Forgetting the unit

In the category of groupoids

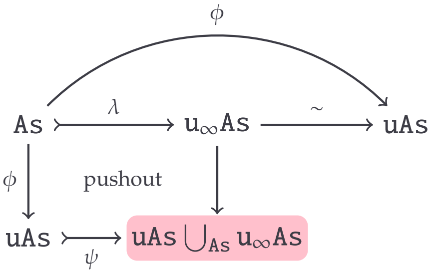
What are algebras over these operads?



Strictly associative monoidal groupoids equipped with an extra strict unit

In the category of groupoids

What are algebras over these operads?

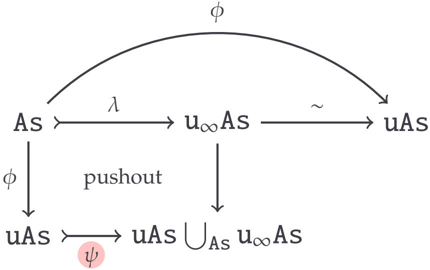


Strictly associative monoidal groupoids
equipped with an extra strict unit

This is the same as a strictly associative and strictly unital monoidal groupoid equipped with an isomorphism $\mathbf{1} \cong I$.

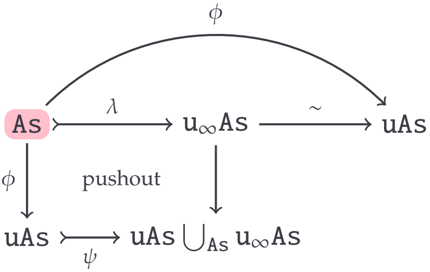
In the category of groupoids

What are algebras over these operads?



IS A WEAK EQUIVALENCE!

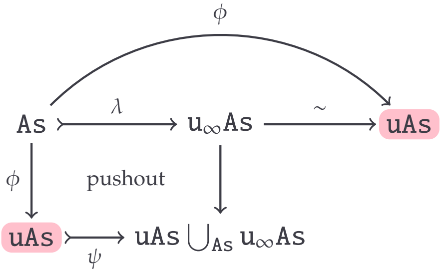
In the category of chain complexes



Generated by

in degree 0 with trivial differential and relation

In the category of chain complexes



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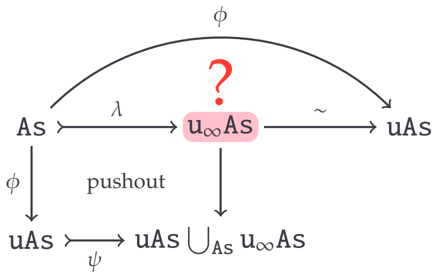
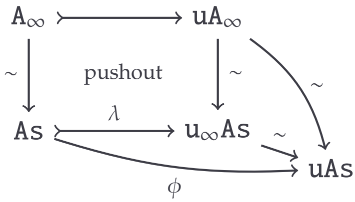


In the category of chain complexes

By relative left properness, λ can be obtained as



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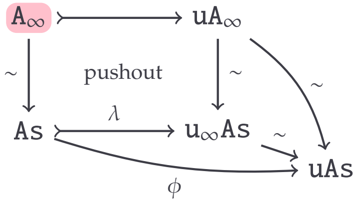
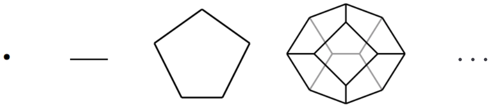


Stasheff's operad, cellular chains on associahedra



In the category of chain complexes

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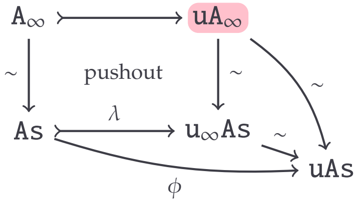
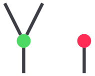
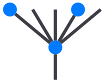


Fukaya–Oh–Ohta–Ono's operad, cellular chains on unital associahedra [M.–Tonks'14]

In the category of chain complexes

The operad $\mathtt{u}_\infty\mathtt{As}$ is generated by

 and all corked corollas like 

The first two generators have degree 0 and trivial differential. The degree of a corked corolla is

$$2 \cdot \#\{\text{corks}\} + \#\{\text{leaves}\} - 2.$$

The only relation is the one in $\mathtt{As}$, hence the inclusion

$$\lambda \colon \mathtt{As} \rightarrowtail \mathtt{u}_\infty\mathtt{As}$$

is a cofibration. The ideal generated by corked corollas is contractible and its quotient is $\mathtt{uAs}$, so

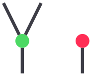
$$\mathtt{u}_\infty\mathtt{As} \xrightarrow{\sim} \mathtt{uAs}.$$

In the category of chain complexes

We define the exhaustive filtration by cofibrations

$$\mathtt{As} = \mathtt{u}_0\mathtt{As} \subset \cdots \subset \mathtt{u}_n\mathtt{As} \subset \cdots \subset \mathtt{u}_\infty\mathtt{As}$$

where, for $n \geq 1$, $\mathtt{u}_n\mathtt{As}$ is the suboperad of $\mathtt{u}_n\mathtt{As}$ generated by

 and all corked corollas with $\leq n$ corks.

In the category of chain complexes

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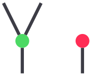 and all corked corollas with $\leq n$ corks.

Lemma

The inclusion $\mathtt{uAs} \bigcup_{\mathtt{As}} \mathrm{u}_{n-1}\mathtt{As} \subset \mathtt{uAs} \bigcup_{\mathtt{As}} \mathrm{u}_n\mathtt{As}$ is always a weak equivalence.

In particular $\psi\colon \mathtt{uAs} = \mathtt{uAs} \bigcup_{\mathtt{As}} \mathrm{u}_0\mathtt{As} \subset \mathtt{uAs} \bigcup_{\mathtt{As}} \mathrm{u}_\infty\mathtt{As}$ too.

By induction on n and relative left properness,

$$\begin{array}{ccc}
\mathtt{uAs} \bigcup_{\mathtt{As}} \mathtt{u}_{n-1}\mathtt{As} & \rightarrowtail & \mathtt{uAs} \bigcup_{\mathtt{As}} \mathtt{u}_{n}\mathtt{As} \\
\sim \downarrow & \text{pushout} & \downarrow \sim \\
\mathtt{uAs} & \rightarrowtail & \mathtt{Q}
\end{array}$$

In the category of chain complexes

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The RED RETRACTION is defined by

By induction on n and relative left properness,

$$\begin{array}{ccc} \mathtt{uAs} \bigcup_{\mathtt{As}} \mathtt{u}_{n-1}\mathtt{As} & \rightarrowtail & \mathtt{uAs} \bigcup_{\mathtt{As}} \mathtt{u}_n\mathtt{As} \\ \sim \downarrow & \text{pushout} & \downarrow \sim \\ \mathtt{uAs} & \rightarrowtail & \mathtt{Q} \end{array}$$

For $n = 1$, the operad $\mathtt{Q}$ is generated by

and all corked corollas with 1 cork.

The only relations are the two ones in $\mathtt{uAs}$.

In the category of chain complexes

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For $n = 1$, the operad $\mathtt{Q}$ is generated by

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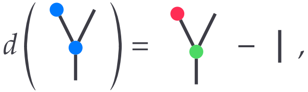

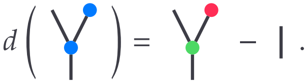

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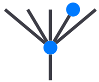
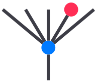
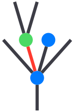
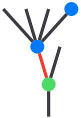
$d\Big($  $\Big) = \pm$ terms of the form , , .

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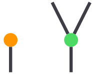
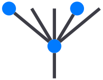
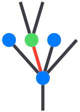
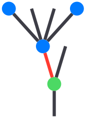
For $n > 1$, the operad $\mathtt{Q}$ is generated by



and all corked corollas with n corks.

The only relations are the two ones in $\mathtt{uAs}$. The differential is

$d\Big($  $\Big) = \pm$ terms of the form  ,  .

In the category of chain complexes

Theorem

Any DG-operad of the form $\mathtt{P} = (\mathtt{F}(S), d)$ has a cylinder

$$\mathtt{IP} = (\mathtt{F}(i_0 S \amalg \Sigma S \amalg i_1 S), d)$$

such that $i_0, i_1 : \mathtt{P} \to \mathtt{IP}$ are DG-maps and

$$d(\Sigma x) = i_0 x - i_1 x + \text{extra terms.}$$

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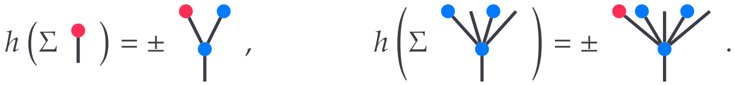
$\mathtt{Q}$ is free and linear relative to $\mathtt{uAs}$ and there is a strong deformation retraction

$$\mathtt{uAs} \rightleftarrows \mathtt{Q} \circlearrowleft h$$



Transfer to other categories

Transfer to other categories

Theorem

Any weak symmetric monoidal Quillen pair $\mathcal{V} \rightleftarrows \mathcal{W}$ induces a Quillen pair $\mathrm{Op}_{\mathcal{V}} \rightleftarrows \mathrm{Op}_{\mathcal{W}}$ and the derived left Quillen functor in $\mathrm{Ho}\,\mathrm{Op}_{\mathcal{V}} \rightleftarrows \mathrm{Ho}\,\mathrm{Op}_{\mathcal{W}}$ preserves $\phi\colon \mathtt{As} \to \mathtt{uAs}$.

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$$\mathrm{Ch}(k) \leftrightarrows \mathrm{Ch}(k)_{\geq 0} \rightleftarrows \mathrm{Mod}(k)^{\Delta^{\mathrm{op}}} \leftrightarrows \mathrm{Set}^{\Delta^{\mathrm{op}}} \rightleftarrows \mathrm{Grd}\,.$$

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Inclusion $\dashv$ truncation.

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Dold–Kan equivalence [Schwede–Shipley'03].

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Free module ⊣ forgetful, fundamental groupoid ⊣ nerve.

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Free module ⊣ forgetful, fundamental groupoid ⊣ nerve.

In simplicial sets, the map $\psi\colon \mathtt{uAs} \to \mathtt{uAs} \bigcup_{\mathtt{As}} \mathtt{u}_\infty\mathtt{As}$ induces an equivalence on fundamental groupoids and a quasi-isomorphism in homology, so it is a weak equivalence.

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$$\mathrm{Set}^{\Delta^{\mathrm{op}}} \rightleftarrows \mathrm{Spectra}$$

Infinite suspension $\dashv$ 0^{th} term.

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$$\mathcal{V} \leftrightarrows \mathrm{Set}^{\Delta^{\mathrm{op}}}.$$

Simplicial $\mathcal{V}$.

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$$\mathcal{V} \leftrightarrows \mathrm{Ch}(k).$$

Complicial $\mathcal{V}$.

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$$\mathcal{V} \leftrightarrows \mathrm{Spectra}\,.$$

Spectral $\mathcal{V}$.

HOW MANY UNITS MAY AN A_∞-ALGEBRA HAVE?

Workshop in Category Theory and Algebraic Topology, Louvain-la-Neuve, 10–12 September 2015.

Fernando Muro

Universidad de Sevilla