UNIQUENESS OF A_∞-STRUCTURES ON GRADED ALGEBRAS WITH RICH HOCHSCHILD COHOMOLOGY

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An old question

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Theorem (Kadeishvili'88)

If $HH^{n,2-n}(H^*(A)) = 0$, $n \geq 3$, then A is formal.

Formality and Massey products

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is a coset of

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If A is formal, then always

$$0 \in \langle x, y, z \rangle.$$

Kadeishvili showed that any differential graded algebra A is determined by a MINIMAL A_∞-ALGEBRA structure on its cohomology

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The ternary operation m_3 is a Hochschild cocycle and

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If A is formal $\{m_3\} = 0$ too.

Massey products and triangulated categories

A TRIANGULATED CATEGORY $\mathcal{T}$ with suspension functor Σ gives rise to a graded category

$$\mathcal{T}^n(X,Y) = \mathcal{T}(X,\Sigma^n X)$$

equipped with a Massey product operation such that exact triangles

$$X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{q} \Sigma X$$

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In this context formality implies that all exact triangles split, which seldom happens.

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$HH^{\bullet,\star}(H^*(A))$ is a commutative algebra.

Theorem

Suppose

$$\begin{aligned} HH^{s,t}(H^*(A)) &\longrightarrow HH^{s+3,t-1}(H^*(A)) \\ x &\mapsto \{m_3\} \cdot x \end{aligned}$$

is an isomorphism for $s \geq 2$. Then A is uniquely determined up to quasi-isomorphism by $H^*(A)$ and $\{m_3\} \in HH^{3,-1}(H^*(A))$.

An application

Corollary

Differential graded models for locally finite triangulated categories $\mathcal{T}$ are uniquely determined by their universal Massey product.

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Differential graded models for locally finite triangulated categories $\mathcal{T}$ are uniquely determined by their universal Massey product.

- Bounded derived categories of algebras of finite representation type.
- Stable module categories of self-injective algebras of finite representation type.
- Cluster categories.

The classifying space

$$B\mathcal{D} = B\mathcal{A}_\infty = \lim B\mathcal{A}_n \to \cdots \to B\mathcal{A}_{n+1} \longrightarrow B\mathcal{A}_n \to \cdots \to B\mathcal{A}_1 = B\mathcal{C}$$

$$\begin{aligned}
\mathcal{D} &= \text{category of differential graded algebras} \\
\mathcal{A}_n &= \text{category of } A_n\text{-algebras } (1 \le n \le \infty) \\
\mathcal{C} &= \text{category of cochain complexes} \\
B\mathcal{M} &= \text{classifying space of a model category } \mathcal{M} \\
&= \text{nerve of the category of weak equivalences in } \mathcal{M}
\end{aligned}$$

The classifying space

A fixed base point $A \in B\mathcal{D}$ allows for the construction of the Bousfield–Kan'72 FRINGED SPECTRAL SEQUENCE of the tower,

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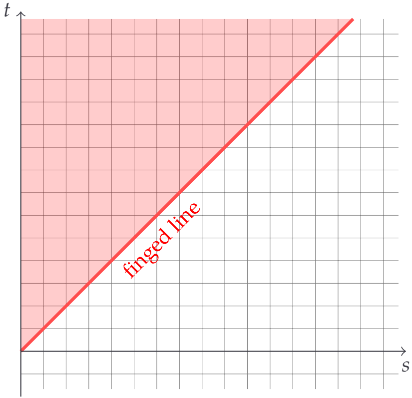
Bousfield–Kan's fringed spectral sequence

$$E_2^{s,t} \underset{s>0}{=} HH^{s+1,t-1}(H^*(A)) \Longrightarrow \pi_{t-s}(B\mathcal{D}, A)$$



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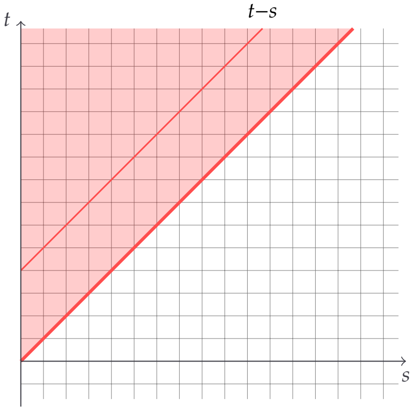

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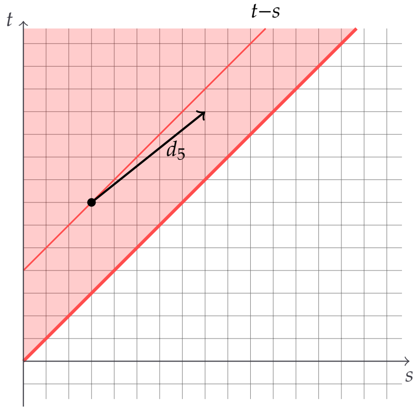

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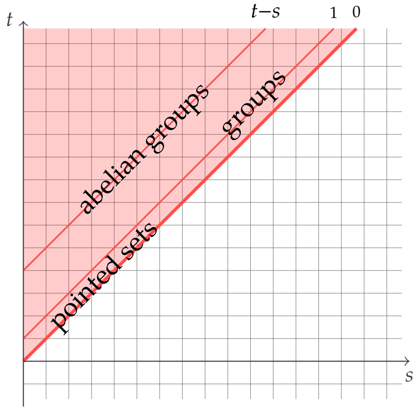
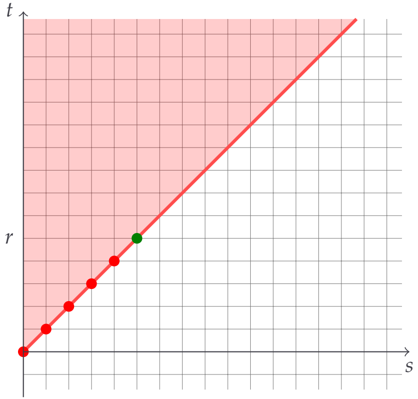


The fringed line and uniqueness

$E_r^{s,s}$ = weak equivalence classes of A_{s+1}-algebras which extend to A_{s+r}-algebras and restrict to the same A_s-algebra as A, $s \leq r$.



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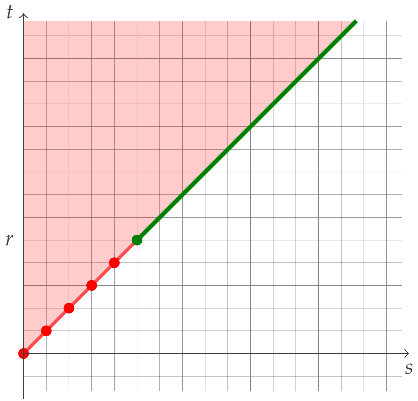
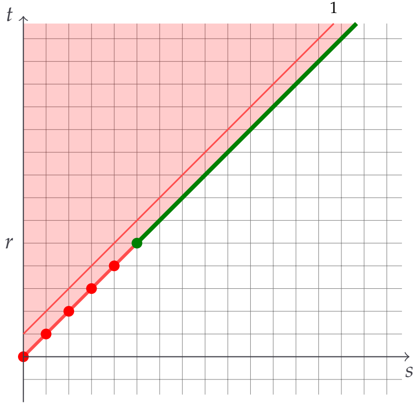


If the green line vanishes, the A_r-algebra underlying A extends to an A_n-algebra for all $n \geq r$ in an essentially unique way.

The fringed line and uniqueness

The obstruction to A_∞-uniqueness is the $\lim^1$ in the Milnor s.e.s.

$$\lim_n{}^1 \pi_1(B\mathcal{A}_n, A) \hookrightarrow \pi_0(B\mathcal{D}, A) \twoheadrightarrow \lim_n \pi_0(B\mathcal{A}_n, A)$$



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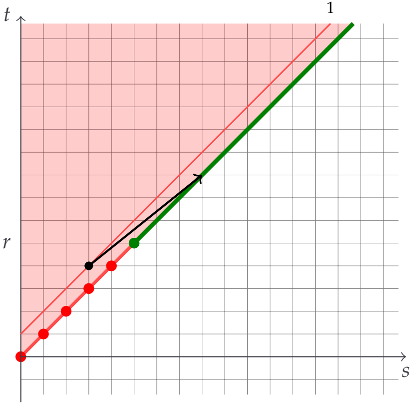

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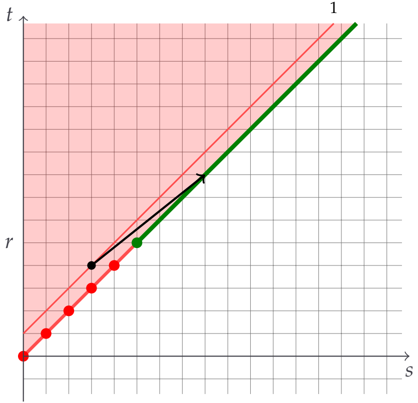
The fringed line and uniqueness

If the green half-line vanishes, $E_r^{s,s} = 0$, $s \geq r$, then A is uniquely determined by its underlying A_r-algebra.



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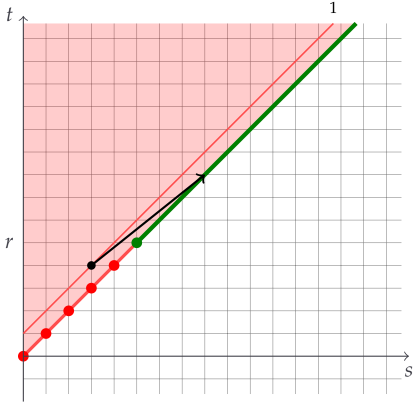
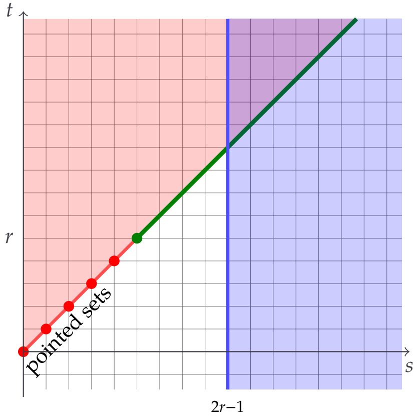


We'd like to show that $E_3^{s,s} = 0$ for $s \geq 3$, since the A_3-algebra underlying A is determined by $\{m_3\} \in HH^{3,-1}(H^*(A))$.

The extended spectral sequence

We have extended the spectral sequence to the blue region in such a way that $E_2^{s,t} = HH^{s+1,t-1}(H^*(A))$ for $s > 0$ where defined.



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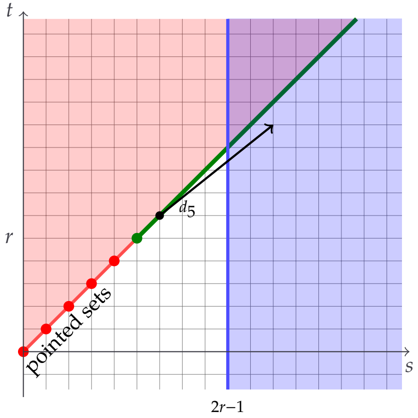

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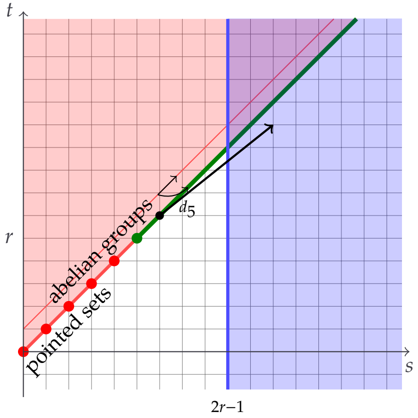

It consists of vector spaces in the blue region and in $t - s \geq 2$.

Beyond the second page

$HH^{\bullet,\star}(H^*(A))$ is a commutative algebra and a Lie algebra in a compatible way (Gerstenhaber algebra).

Theorem

Recall that $E_2^{s,t} = HH^{s+1,1-t}(H^*(A))$ for $s > 0$. The second differential is the Lie bracket with the universal Massey product,

$$d_2 = [\{m_3\}, -]\colon HH^{s+1,1-t}(H^*(A)) \longrightarrow HH^{s+3,-t}(H^*(A)).$$

Beyond the second page

The product by the EULER CLASS $\{\delta\} \in HH^{1,0}(H^*(A))$,
$\delta(x) = |x| \cdot x$ is a nullhomotopy for the product by $\{m_3\}$,

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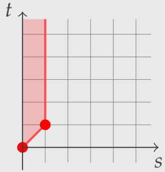
Proposition

If the following map is an isomorphism for $s \geq 2$, then E_3 is concentrated in $s = 0, 1$,

$$\begin{aligned} HH^{s,t}(H^*(A)) &\longrightarrow HH^{s+3,t-1}(H^*(A)) \\ x &\mapsto \{m_3\} \cdot x \end{aligned}$$



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$\mathtt{A}_n$ = operad for A_n-algebras.

Proposition

For $1 \leq s \leq m \leq r$, there is a linear $\mathtt{A}_m$-bimodule $\mathtt{B}_{m,r,s}$ and a cofiber sequence in the homotopy category of operads rel. $\mathtt{A}_m$

$$\mathtt{F}_{\mathtt{A}_m}(\Sigma^{-1}_{\mathtt{A}_m}\mathtt{B}_{m,r,s}) \to \mathtt{A}_r \rightarrowtail \mathtt{A}_{r+s}.$$

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Given an operad $\mathtt{P} = \{\mathtt{P}(n)\}_{n\ge 0}$, a LINEAR $\mathtt{P}$-MODULE $\mathtt{B}$ is a sequence $\mathtt{B} = \{\mathtt{B}(n)\}_{n\ge 0}$ equipped with maps, $1 \le i \le s$,

$$\mathtt{P}(s) \otimes \mathtt{B}(t) \xrightarrow{\circ_i} \mathtt{B}(s+t-1) \xleftarrow{\circ_i} \mathtt{B}(s) \otimes \mathtt{P}(t)$$

satisfying the obvious associativity and unitality laws, e.g. $\mathtt{B} = \mathtt{P}$.

The category of linear $\mathtt{P}$-modules is a pointed stable $\mathcal{C}$-model category and there is a Quillen pair

$$\text{linear } \mathtt{P}\text{-modules} \overset{\mathtt{F}_\mathtt{P}}{\rightleftarrows} \mathtt{P} \downarrow \text{Operads}.$$

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