Triangulated categories

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Let **A** be an abelian category, e.g. $\mathbf{A} = \text{Mod-}R$, right modules over a ring R.

The category $\mathbf{C}(\mathbf{A})$ of complexes in **A**,

$$X = \{\cdots \to X_{n-1} \xrightarrow{d} X_n \xrightarrow{d} X_{n+1} \to \cdots\} \quad (d^2 = 0),$$

is also abelian.

Definition

A morphism $f\colon X \xrightarrow{\sim} Y$ in $\mathbf{C}(\mathbf{A})$ is a quasi-isomorphism if it induces isomorphisms in cohomology,

$$H^n(f)\colon H^n(X) \xrightarrow{\cong} H^n(Y), \quad n \in \mathbb{Z}.$$

The derived category

Let $\mathbf{A}$ be an abelian category, e.g. $\mathbf{A} = \text{Mod-}R$, right modules over a ring R.

The category $\mathbf{C}(\mathbf{A})$ of complexes in $\mathbf{A}$,

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Example

If P and I are a projective and an injective resolution of M in $\mathbf{A}$, respectively, then we have quasi-isomorphisms,

$$
\begin{array}{ccccccccccccccc}
P & & \cdots & \longrightarrow & P_{-2} & \longrightarrow & P_{-1} & \longrightarrow & P_0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow \wr & & & & \downarrow & & \downarrow & & \downarrow\!\!\downarrow & & \downarrow & & \downarrow & & \\
M & & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow \wr & & & & \downarrow & & \downarrow & & \hookdownarrow & & \downarrow & & \downarrow & & \\
I & & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & I_0 & \longrightarrow & I_1 & \longrightarrow & I_2 & \longrightarrow & \cdots
\end{array}
$$

The derived category

Definition

The derived category $\mathbf{D}(\mathbf{A})$ *is a category equipped with a functor*

$$p\colon \mathbf{C}(\mathbf{A}) \longrightarrow \mathbf{D}(\mathbf{A})$$

such that:

- *p takes quasi-isomorphisms to isomorphisms,*
- *p is universal among the functors satisfying this property, i.e. if $p'\colon \mathbf{C}(\mathbf{A}) \to \mathbf{B}$ takes quasi-isomorphisms to isomorphisms then there exists a unique functor $p''\colon \mathbf{D}(\mathbf{A}) \to \mathbf{B}$ such that $p' = p''p$,*

$$\begin{array}{ccc} \mathbf{C}(\mathbf{A}) & \xrightarrow{\;p\;} & \mathbf{D}(\mathbf{A}) \\ & {\scriptstyle p'}\searrow & \downarrow{\scriptstyle \exists!\, p''} \\ & & \mathbf{B} \end{array}$$

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Question: What's the algebraic structure of $\mathbf{D}(\mathbf{A})$?

Answer: Triangulated category!

Remark

- *The derived category need not exist [Freyd'64].*
- *If it exists then it is uniquely defined up to isomorphism.*
- *An object M in $\mathbf{A}$ becomes isomorphic to any projective resolution in $\mathbf{D}(\mathbf{A})$, and also to any injective resolution.*
- *The cohomology functor factors through the derived category,*

$$\begin{array}{ccc} \mathbf{C}(\mathbf{A}) & \xrightarrow{p} & \mathbf{D}(\mathbf{A}) \\ & \searrow^{H^*} & \downarrow{\scriptstyle \exists !} \\ & & \mathbf{A}^{\mathbb{Z}} \end{array}$$

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Example

- *If k is a field, the previous cohomology functor*

$$H^*\colon \mathbf{D}(\text{Mod-}k) \xrightarrow{\simeq} (\text{Mod-}k)^{\mathbb{Z}}$$

  is an equivalence of categories.

- *If R is a hereditary ring, such as $\mathbb{Z}$, $k[X]$, or the path algebra of a quiver, then the functor*

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- *One can similarly define the derived category* $\mathbf{D}(\mathbf{E})$ *of an exact category* $\mathbf{E} \subset \mathbf{A}$*, in this case cohomology is a functor*

$$H^*\colon \mathbf{C}(\mathbf{E}) \longrightarrow \mathbf{A}^{\mathbb{Z}}.$$

- *One can also define the derived category of a differential graded algebra* A*, denoted by* $\mathbf{D}(A)$*, replacing the category of complexes with* Mod-A*, for which the cohomology functor is*

$$H^*\colon \text{Mod-}A \longrightarrow \text{Mod-}H^*(A).$$

- *One can more generally consider differential graded categories, a.k.a. DGAs with several objects.*

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Definition

A morphism $f\colon X \to Y$ in $\mathbf{C}(\mathbf{A})$ is nullhomotopic $f \simeq 0$ if there exist morphisms, called the homotopy,

$$h\colon X_n \longrightarrow Y_{n-1}, \quad n \in \mathbb{Z},$$

such that

$$f = hd + dh.$$

The homotopy category $\mathbf{K}(\mathbf{A})$ is the quotient of $\mathbf{C}(\mathbf{A})$ by the ideal of nullhomotopic morphisms.

Two morphisms $f, g\colon X \to Y$ in $\mathbf{C}(\mathbf{A})$ are homotopic $f \simeq g$ if $f - g$ is nullhomotopic.

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The homotopy category

The homotopy category approaches the derived category.

Proposition

Two homotopic morphisms in $\mathbf{C}(\mathbf{A})$ *map to the same morphism in the derived category* $\mathbf{D}(\mathbf{A})$. *In particular there is a factorization*

$$\begin{array}{ccc} \mathbf{C}(\mathbf{A}) & \twoheadrightarrow & \mathbf{K}(\mathbf{A}) \\ & {\scriptstyle p} \searrow & \downarrow {\scriptstyle \exists !} \\ & & \mathbf{D}(\mathbf{A}) \end{array}$$

The algebraic structure of $\mathbf{K}(\mathbf{A})$ is also that of a triangulated category. We will construct $\mathbf{D}(\mathbf{A})$ from $\mathbf{K}(\mathbf{A})$.

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Exact triangles

Definition

The *mapping cone* *of a morphism* $f\colon X \to Y$ *in* $\mathbf{C}(\mathbf{A})$ *is the complex* C_f *with*

$$(C_f)_n = Y_n \oplus X_{n+1}$$

and differential

$$d_{C_f}\colon (C_f)_{n-1} = Y_{n-1} \oplus X_n \xrightarrow{\begin{pmatrix} d_Y & f \\ 0 & -d_X \end{pmatrix}} Y_n \oplus X_{n+1} = (C_f)_n.$$

The *suspension* *or* *shift* ΣX *of* X *in* $\mathbf{C}(\mathbf{A})$ *is the mapping cone of the trivial morphism* $0 \to X$*, i.e.* $(\Sigma X)_n = X_{n+1}$, $d_{\Sigma X} = -d_X$.

The obvious sequence of morphisms in $\mathbf{C}(\mathbf{A})$,

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{q} \Sigma X,$$

is called an *exact triangle* *when mapped to* $\mathbf{K}(\mathbf{A})$ *or* $\mathbf{D}(\mathbf{A})$.

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Question: Where do short exact sequences in $\mathbf{C}(\mathbf{A})$ go in $\mathbf{D}(\mathbf{A})$?

Proposition

Given a short exact sequence $X \overset{f}{\hookrightarrow} Y \overset{g}{\twoheadrightarrow} Z$ in $\mathbf{C}(\mathbf{A})$ there is a quasi-isomorphism $C_f \overset{\sim}{\to} Z$ defined by

$$(C_f)_n = Y_n \oplus X_{n+1} \xrightarrow{\binom{g}{0}} Z_n, \quad n \in \mathbb{Z},$$

and the following diagram commutes in $\mathbf{C}(\mathbf{A})$,

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{i} & C_f & \xrightarrow{q} & \Sigma X \\ & & & {\searrow}^{g} & \downarrow{\wr} & & \\ & & & & Z & & \end{array}$$

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Definition

A *suspended category* *is a pair* $(\mathbf{T}, \Sigma)$ *given by:*

- *An additive category* $\mathbf{T}$*.*
- *A self-equivalence* $\Sigma\colon \mathbf{T} \xrightarrow{\simeq} \mathbf{T}$ *called* *suspension* *or* *shift*.

A *triangle* *in* $(\mathbf{T}, \Sigma)$ *is a diagram of the form*

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Here f *is called the* *base*. *This diagram can also be depicted as*

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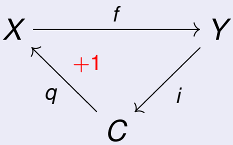

Definition

A ***morphism of triangles*** *in* $(\mathbf{T}, \Sigma)$ *is a commutative diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{i} & C & \xrightarrow{q} & \Sigma X \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{i'} & C' & \xrightarrow{q'} & \Sigma X' \end{array}$$

Triangulated categories

Definition (Puppe, Verdier'60s)

A triangulated category *is a triple* $(\mathbf{T}, \Sigma, \triangle)$ *consisting of a suspended category* $(\mathbf{T}, \Sigma)$ *and a class of triangles* $\triangle$*, called* exact triangles*, satisfying the following four axioms:*

TR1 *The class* $\triangle$ *is closed by isomorphisms, every morphism* $f\colon X \to Y$ *in* $\mathbf{T}$ *is the base of an exact triangle*

$$X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{q} \Sigma X,$$

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is always exact.

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TR2 *A triangle*

$$X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{q} \Sigma X$$

is exact if and only if its translation

$$Y \xrightarrow{i} C \xrightarrow{q} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is exact.

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TR3 *Any commutative square between the bases of two exact triangles can be completed to a morphism of triangles*

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If $(\mathbf{T}, \Sigma, \triangle)$ satisfies just these three axioms we say that it is a Puppe triangulated category.

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Example (TR3 for $\mathbf{K}(\mathbf{A})$)

In the homotopy category $\mathbf{K}(\mathbf{A})$,

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\end{array}$$

We choose representatives of these homotopy classes, that we denote by the same name.

Let $h\colon X_{n+1} \to Y'_n$, $n \in \mathbb{Z}$, *be a homotopy* $\beta f \simeq f'\alpha$. *Define*

$$\gamma\colon (C_f)_n = Y_n \oplus X_{n+1} \xrightarrow{\begin{pmatrix} \beta & h \\ 0 & \alpha \end{pmatrix}} Y'_n \oplus X'_{n+1} = (C_{f'})_n.$$

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X' & \xrightarrow{f'} & Y' & \xrightarrow{i'} & C_{f'} & \xrightarrow{q'} & \Sigma X'
\end{array}$$

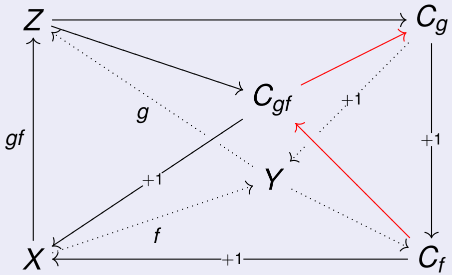
We choose representatives of these homotopy classes, that we denote by the same name.

Let $h\colon X_{n+1} \to Y'_n$, $n \in \mathbb{Z}$, *be a homotopy* $\beta f \simeq f'\alpha$. *Define*

$$\gamma\colon (C_f)_n = Y_n \oplus X_{n+1} \xrightarrow{\left(\begin{smallmatrix}\beta & h\\ 0 & \alpha\end{smallmatrix}\right)} Y'_n \oplus X'_{n+1} = (C_{f'})_n.$$

Definition (Verdier's octahedral axiom)

TR4 *Given two composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbf{T}$, and three exact triangles with bases f, g and gf,*



there are morphisms in red completing the diagram commutatively in such a way that the front right triangle is exact.

Triangulated functors

Definition

A ***triangulated functor***

$$(F,\phi)\colon (\mathbf{T},\Sigma,\triangle)\longrightarrow(\mathbf{T}',\Sigma',\triangle')$$

consists of an additive functor $F\colon \mathbf{T}\to\mathbf{T}'$ *together with a natural isomorphism* $\phi\colon F\Sigma\cong\Sigma' F$ *such that for any exact triangle in the source*

$$X\stackrel{f}{\longrightarrow}Y\stackrel{i}{\longrightarrow}C\stackrel{q}{\longrightarrow}\Sigma X$$

the image triangle

$$F(X)\stackrel{F(f)}{\longrightarrow}F(Y)\stackrel{F(i)}{\longrightarrow}F(C)\stackrel{\phi(X)F(q)}{\longrightarrow}\Sigma' F(X)$$

is exact in the target.

Remark

- *There is no known Puppe triangulated category which does not satisfy the octahedral axiom.*
- *Any triangulated structure on $(\mathbf{T}, \Sigma)$ induces a triangulated structure on $(\mathbf{T}^{\mathrm{op}}, \Sigma^{-1})$.*
- *The third object C in an exact triangle $X \xrightarrow{f} Y \xrightarrow{i} C \to \Sigma X$, which is called the mapping cone of f, is well defined by f up to non-canonical isomorphism.*

Definition

A full additive subcategory $\mathbf{S} \subset \mathbf{T}$ is a triangulated subcategory if Σ restricts to a self-equivalence in $\mathbf{S}$ and the mapping cone in $\mathbf{T}$ of any morphism in $\mathbf{S}$ lies in $\mathbf{S}$.

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Example

We can consider the following triangulated subcategories of $\mathbf{K}(\mathbf{A})$*:*

- $\mathbf{K}^{+}(\mathbf{A})$, *formed by bounded below complexes,*

$$\cdots \to 0 \longrightarrow X_n \stackrel{d}{\longrightarrow} X_{n+1} \to \cdots .$$

- $\mathbf{K}^{-}(\mathbf{A})$, *formed by bounded above complexes,*

$$\cdots \to X_{n-1} \stackrel{d}{\longrightarrow} X_n \longrightarrow 0 \to \cdots .$$

- $\mathbf{K}^{b}(\mathbf{A})$, *formed by bounded complexes,*

$$\cdots \to 0 \longrightarrow X_n \to \cdots \to X_{n+m} \longrightarrow 0 \to \cdots .$$

Verdier quotients

Definition

Let $\mathbf{T}$ *be a triangulated category. We say that a triangulated subcategory* $\mathbf{S} \subset \mathbf{T}$ *is* ***thick*** *if it contains all the direct summands of its objects.*

The ***Verdier quotient*** $\mathbf{T}/\mathbf{S}$ *is a triangulated category equipped with a triangulated functor*

$$\mathbf{T} \longrightarrow \mathbf{T}/\mathbf{S}$$

which is universal among those taking the objects in $\mathbf{S}$ *to zero objects.*

Example

The triangulated subcategory $\mathbf{Ac}(\mathbf{A}) \subset \mathbf{K}(\mathbf{A})$ *formed by the complexes* X *with trivial cohomology* $H^*(X) = 0$*, called* ***acyclic****, is thick.*

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Verdier quotients

Theorem

The functor

$$\mathbf{C}(\mathbf{A}) \twoheadrightarrow \mathbf{K}(\mathbf{A}) \longrightarrow \mathbf{K}(\mathbf{A})/\mathbf{Ac}(\mathbf{A})$$

satisfies the universal property of the derived category, i.e.

$$\mathbf{D}(\mathbf{A}) = \mathbf{K}(\mathbf{A})/\mathbf{Ac}(\mathbf{A}),$$

in particular the derived category is triangulated with the structure defined above.

...and similarly for exact categories and DGAs (possibly with several objects).

The Verdier quotient $\mathbf{T}/\mathbf{S}$ can be explicitly constructed as follows:

- Objects in $\mathbf{T}/\mathbf{S}$ are the same as in $\mathbf{T}$.
- A morphism in $(\mathbf{T}/\mathbf{S})(X, Y)$ is represented by a diagram in $\mathbf{T}$

$$X \overset{f}{\longleftarrow} A \overset{g}{\longrightarrow} Y,$$

  where the mapping cone of f is in $\mathbf{S}$.
- Another such diagram

$$X \overset{f'}{\longleftarrow} A' \overset{g'}{\longrightarrow} Y$$

  represents the same morphism in $\mathbf{T}/\mathbf{S}$ if there is a commutative diagram in $\mathbf{T}$,

$$\begin{array}{ccccc} & & A & & \\ & \swarrow^{f} & \downarrow & \searrow^{g} & \\ X & & & & Y \\ & \nwarrow_{f'} & A' & \nearrow_{g'} & \end{array}$$

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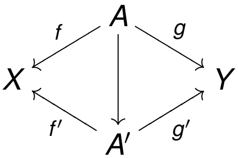
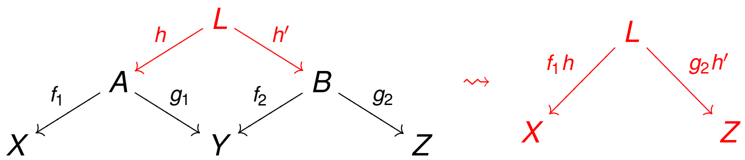


Verdier quotients

- The equivalence relation generated by the previous relation defines morphism sets in $\mathbf{T}/\mathbf{S}$.
- The composition of two morphisms in $\mathbf{T}/\mathbf{S}$ in terms of representatives is done as follows:



such that there is an exact triangle in $\mathbf{T}$,

$$L \xrightarrow{\binom{-h}{h'}} A \oplus B \xrightarrow{(g_1\ f_2)} Y \longrightarrow \Sigma L.$$

Verdier quotients

- The suspension in $\mathbf{T}/\mathbf{S}$ is defined by the suspension Σ in $\mathbf{T}$ on objects and diagrams representing morphisms,

$$\Sigma(X \xleftarrow{f} A \xrightarrow{g} Y) = \Sigma X \xleftarrow{\Sigma f} \Sigma A \xrightarrow{\Sigma g} \Sigma Y.$$

- The universal functor $(F, \phi)\colon \mathbf{T} \to \mathbf{T}/\mathbf{S}$ is the identity on objects $F(X) = X$ and it is defined on morphisms as follows:

$$F(f\colon X \to Y) = X \xleftarrow{1_X} X \xrightarrow{f} Y.$$

- The natural transformation $\phi\colon F\Sigma \cong \Sigma F$ is the identity.

- Exact triangles in $\mathbf{T}/\mathbf{S}$ are defined so that they coincide with the triangles isomorphic to the image of the exact triangles in $\mathbf{T}$ by the universal triangulated functor $\mathbf{T} \to \mathbf{T}/\mathbf{S}$. skip remark

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Remark

- *There are triangulated subcategories*

$$\mathbf{D}^b(\mathbf{A}) \subset \mathbf{D}^+(\mathbf{A}), \mathbf{D}^-(\mathbf{A}) \subset \mathbf{D}(\mathbf{A})$$

  as in the homotopy category.
- $\mathbf{A}$ *can be regarded as the full subcategory of complexes concentrated in degree zero in* $\mathbf{D}(\mathbf{A})$.
- *Given* X *and* Y *in* $\mathbf{A}$,

$$\mathbf{D}(\mathbf{A})(X, \Sigma^n Y) = \begin{cases} \mathrm{Ext}^n_{\mathbf{A}}(X, Y), & n \geq 0; \\ 0, & n < 0. \end{cases}$$

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Cohomological functors

Definition

Let $\mathbf{T}$ *be a triangulated category and* $\mathbf{A}$ *an abelian category. A functor* $H\colon \mathbf{T} \to \mathbf{A}$ *is* *cohomological* *if it takes an exact triangle in* $\mathbf{T}$*,*

$$X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{q} \Sigma X,$$

to an exact sequence in $\mathbf{A}$*,*

$$H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(i)} H(C).$$

Cohomological functors

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- *Actually, H takes exact triangles to long exact sequences*

$$\cdots \to H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(i)} H(C) \xrightarrow{H(q)} H(\Sigma X) \xrightarrow{H(\Sigma f)} H(\Sigma Y) \to \cdots.$$

- *The functors*

$$H^0 \colon \mathbf{K}(\mathbf{A}) \longrightarrow \mathbf{A}, \qquad H^0 \colon \mathbf{D}(\mathbf{A}) \longrightarrow \mathbf{A},$$

are cohomological.

- *For any object X in a triangulated category $\mathbf{T}$, the representable functor*

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Definition

Let $\mathbf{T}$ be a triangulated category with coproducts. An object X in $\mathbf{T}$ is compact if $\mathbf{T}(X, -)$ preserves coproducts.

$\mathbf{T}$ is compactly generated if there is a set S of compact objects such that an object Y in $\mathbf{T}$ is trivial iff $\mathbf{T}(X, Y) = 0$ for all $X \in S$.

Example (Neeman'96)

If X is a quasi-compact separated scheme then $D(\mathbf{Qcoh}(X))$ is compactly generated.

Theorem (Brown'62, Neeman'96)

If $\mathbf{T}$ is a compactly generated triangulated category, then any cohomological functor preserving products $H\colon \mathbf{T}^{\mathrm{op}} \to \mathbf{Ab}$ is representable $H = \mathbf{T}(-, Y)$.

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Corollary

Let $F\colon \mathbf{S} \to \mathbf{T}$ be a triangulated functor with compactly generated source. If F preserves coproducts then it has a right adjoint.

Proof.

The right adjoint G must satisfy $\mathbf{S}(-, G(X)) = \mathbf{T}(F(-), X)$. This later functor is well defined and representable by the previous theorem, hence G exists. □

Example (Grothendieck duality)

If $f\colon X \to Y$ is a separated morphism of quasi-compact separated schemes, then the right derived functor of the direct image,

$$\mathbb{R}f_*\colon D(\mathbf{Qcoh}(X)) \longrightarrow D(\mathbf{Qcoh}(Y)),$$

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Theorem (Adams representability theorem, Neeman'97)

If $\mathbf{T}$ *is compactly generated and* $\operatorname{card} \mathbf{T}^c$ *is countable then:*

1. *Every cohomological functor* $H\colon (\mathbf{T}^c)^{\mathrm{op}} \to \mathbf{Ab}$ *is* $H = \mathbf{T}(-,X)_{|\mathbf{S}}$ *for some* X *in* $\mathbf{T}$*.*
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Remark

For instance, $\mathbf{T} = \mathbf{D}(\mathbb{Z})$ *or the stable homotopy category.*

Adams representability

Remark

If $\mathbf{S} \subset \mathbf{T}$ *is a triangulated subcategory. For any object* X *in* $\mathbf{T}$*, the restriction of a representable functor in* $\mathbf{T}$ *is cohomological in* $\mathbf{S}$*,*

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For $\mathbf{T} = \mathbf{D}(\mathbb{C}[x, y])$, *part 1 of Adams representability theorem holds under the continuum hypothesis.*

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An additive functor $F\colon \mathbf{A} \to \mathbf{B}$ induces an obvious triangulated functor

$$F\colon \mathbf{K}(\mathbf{A}) \to \mathbf{K}(\mathbf{B}).$$

If F is exact then it also induces a functor at the level of derived categories,

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If **A** *has enough projectives then the following composite is a triangulated equivalence*

$$\varphi\colon \mathbf{K}^-(\mathrm{Proj}(\mathbf{A})) \xrightarrow{\textit{incl.}} \mathbf{K}^-(\mathbf{A}) \longrightarrow \mathbf{D}^-(\mathbf{A}).$$

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The *left derived functor* *of an additive functor* $F\colon \mathbf{A}\to\mathbf{B}$ *is the composite*

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Remark

The usual left derived functors $\mathbb{L}_nF\colon \mathbf{A}\to\mathbf{B}$ *are recovered as*

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The *right derived functor* *of an additive functor* $F\colon \mathbf{A} \to \mathbf{B}$ *is the composite*

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Suppose that $\mathbf{A}$ has exact coproducts and a projective generator P, e.g. $\mathbf{A} = \text{Mod-}R$ and $P = R$. Let $\mathbf{P} \subset \mathbf{K}(\mathbf{A})$ the smallest triangulated subcategory with coproducts containing P.

Theorem

The composite

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Suppose that **A** has exact products and an injective cogenerator I, e.g. $\mathbf{A} = \text{Mod-}R$ and $I = \text{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$. Let $\mathbf{I} \subset \mathbf{K}(\mathbf{A})$ be the smallest triangulated subcategory with products containing I.

Theorem

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Algebraic triangulated categories

Theorem

With the suspension of complexes and the exact triangles indicated above, the homotopy category $\mathbf{K}(\mathbf{A})$ of an additive category $\mathbf{A}$ is a triangulated category.

Remark

The same result holds for differential graded algebras (possibly with several objects).

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A triangulated category is algebraic if it is triangulated equivalent to a triangulated subcategory of $\mathbf{K}(\mathbf{A})$ for some additive category $\mathbf{A}$.

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Let X be an object in an algebraic triangulated category $\mathbf{T}$ and let

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be an exact triangle, $n \in \mathbb{Z}$. Then

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Proof.

We can directly suppose $\mathbf{T} = \mathbf{K}(\mathbf{A})$. If we take X/n to be the mapping cone of $n \cdot 1_X \colon X \to X$ then it is easy to check that $n \cdot 1_{X/n} \colon X/n \to X/n$ in $\mathbf{C}(\mathbf{A})$ is nullhomotopic. □

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Definition

An abelian category **A** *is* *Frobenius* *if it has enough injectives and projectives, and injective and projective objects coincide.*

Example

- Mod-R *for* R *a* *quasi-Frobenius* *ring, i.e.* R *is right noetherian and right self-injective.*
- *Also* mod-R*, the full subcategory of finitely presented modules.*
- *Examples of quasi-Frobenius rings are fields, division algebras,* $\mathbb{Z}/n$, $k[X]/(f)$*, and the group algebra* kG *of a finite group* G*.*
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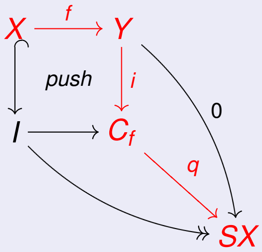

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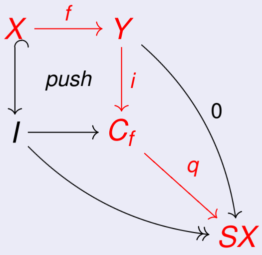

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- *Taking* 0*-cocycles defines a triangulated equivalence*

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$$\text{mod-}R \longrightarrow \mathbf{D}^b(\text{mod-}R) \longrightarrow \mathbf{D}^b(\text{mod-}R)/\mathbf{D}^b(\text{proj-}R)$$

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The suspension functor extends as follows,

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Theorem (Heller’68)

If $\mathbf{T}$ is idempontent complete, the Puppe triangulated structures on $(\mathbf{T}, \Sigma)$ are in bijection with the natural isomorphisms $\theta\colon \Sigma^3 \cong S$ such that

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The ‘moduli space’ of triangulated structures

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Example

Consider the suspended category $(\mathbf{T},\Sigma) = (\text{mod-}k, \textit{identity})$, k *a field.* *In this case* $\text{mod-}\mathbf{T} = \text{mod-}k$ *and* $\underline{\text{mod}}\text{-}k = 0$ *is trivial, hence* $(\mathbf{T},\Sigma)$ *has a unique Puppe triangulated structure.*

As one can easily check, a triangle in $\text{mod-}k$

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \nwarrow & & \swarrow \\ & C & \end{array}$$

is exact iff it is contractible, and $\mathbf{T}$ *satisfies the octahedral axiom.*

It is algebraic, actually there is a triangulated equivalence

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Consider the suspended category $(\mathbf{T}, \Sigma) = (\text{mod-}k, \textit{identity})$*, k a field. In this case* $\text{mod-}\mathbf{T} = \text{mod-}k$ *and* $\underline{\text{mod}}\text{-}k = 0$ *is trivial, hence* $(\mathbf{T}, \Sigma)$ *has a unique Puppe triangulated structure.*

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Theorem (M-Schwede-Strickland'07)

The unique Puppe triangulated structure on proj-$\mathbb{Z}/4$ *with* $\Sigma =$ *the indentity satisfies the octahedral axiom.*

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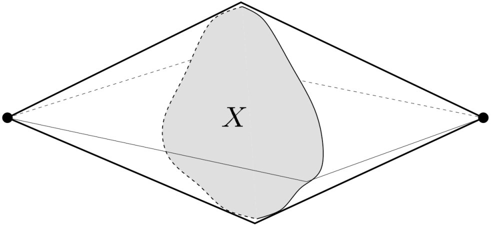
The Spanier-Whitehead category is the triangulated category **SW** defined as:

Obj (X, n), where X is a finite pointed CW-complex and $n \in \mathbb{Z}$.

Map $\mathbf{SW}((X,n),(Y,m)) = \lim_{k\to+\infty} [\Sigma^{k+n}X, \Sigma^{k+m}Y]$.

$\Sigma X \quad =$



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Topological triangulated categories

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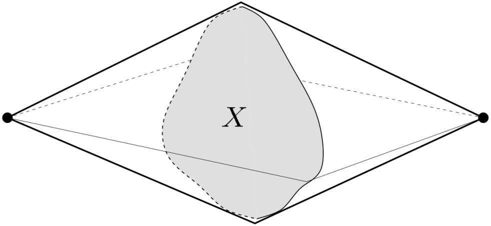

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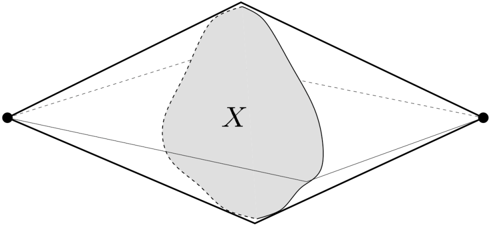

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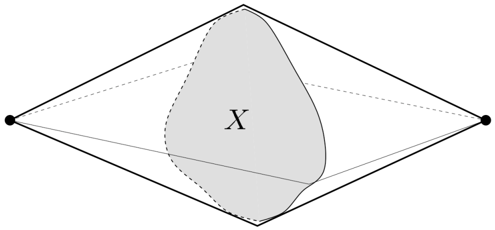
$\Sigma X \;=$ 

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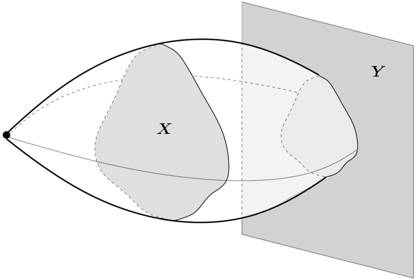

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There is a sequence of pointed maps,

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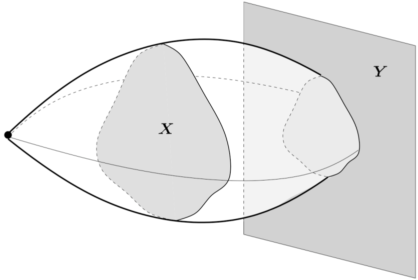

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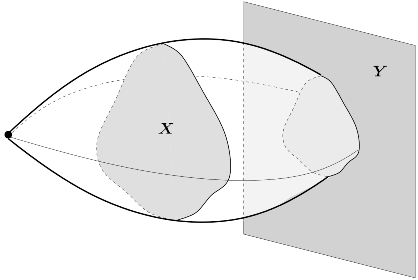

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Example

If $S = (S^0, 0)$ *then there is an exact triangle in* **SW**,

$$S \xrightarrow{2\cdot 1_S} S \xrightarrow{i} S/2 \xrightarrow{q} \Sigma S,$$

where $S/2 = (\mathbb{R}P^2, -1)$. *The map*

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Proposition (Schwede-Shipley'02)

SW *is the 'free topological triangulated category' on one generator S.*
In particular if X is an object in a topological triangulated category **T** *then there is an exact functor*

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We stated above:

Theorem

The unique triangulated structure on proj-$\mathbb{Z}/4$ *with* $\Sigma =$ *the indentity is not topological.*

Proof.

Assume it is topological. Let $F\colon \mathbf{SW} \to$ proj-$\mathbb{Z}/4$ be an exact functor as above for $X = \mathbb{Z}/4$. By the previous example, since $X/2 = X$ in proj-$\mathbb{Z}/4$,

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Remark

There are many different kinds of models for triangulated categories:

- *Stable model categories.*
- *Stable homotopy categories [Heller'88].*
- *Triangulated derivators [Grothendieck'90].*
- *Stable ∞-categories [Lurie'06].*

In all these cases the 'free model in one generator' is associated to the triangulated category **SW***, therefore the exotic triangulated catgory* proj-$\mathbb{Z}/4$ *does not admit any of these kinds of models.*

Moreover, it can neither be obtained out of a triangulated 2-category [Baues-M'08].

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