When can we enhance a triangulated category?

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Outline

When can we enhance a triangulated category $\mathcal{T}$?

- When is $\mathcal{T}$ algebraic?
  - Over a field k.
  - Over an arbitrary commutative ring k.
- When is $\mathcal{T}$ topological?

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Algebraic triangulated categories

Definition (Keller'06)

A triangulated category $\mathcal{T}$ is algebraic if $\mathcal{T} \simeq \underline{\mathcal{E}}$ is equivalent to the stable category $\underline{\mathcal{E}}$ of a Frobenius exact category $\mathcal{E}$.

Theorem (Keller'94)

If $\mathcal{T}$ is compactly generated then $\mathcal{T}$ is algebraic $\Leftrightarrow$ $\mathcal{T} = H^0(\mathcal{A})$ for a pretriangulated A_∞-category $\mathcal{A}$.

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A_∞-categories

Definition

An A_∞-category $\mathcal{A}$ consists of

- *Objects $X, Y, \ldots$*
- *Morphism $\mathbb{Z}$-graded k-modules $\mathcal{A}(X, Y)$,*
- *Identity morphisms $\mathrm{id}_X \in \mathcal{A}(X, X)^0$,*
- *n-Fold composition laws, $n \geq 1$,*

$$m_n \colon \mathcal{A}(X_{n-1}, X_n) \otimes \cdots \otimes \mathcal{A}(X_0, X_1) \longrightarrow \mathcal{A}(X_0, X_n),$$

$$\deg(m_n) = 2 - n.$$

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Composition laws must satisfy equations:

- $m_1 m_1 = 0$, i.e. $\mathcal{A}(X, Y)$ are complexes.
- $m_1 m_2 = m_2(1 \otimes m_1 + m_1 \otimes 1)$, i.e. m_1 is a derivation for the product m_2.
- $m_2(m_2 \otimes 1 - 1 \otimes m_2) = m_1 m_3 + m_3(1 \otimes 1 \otimes m_1 + 1 \otimes m_1 \otimes 1 + m_1 \otimes 1 \otimes 1)$, i.e. m_2 is associative up to the homotopy m_3.
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Example

- *One-object A_∞-categories are Stasheff's A_∞-algebras.*
- *DG-categories are A_∞-categories with $m_n = 0$ for all $n \geq 3$.*

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Minimal A_∞-categories

Definition

An A_∞-category $\mathcal{A}$ is minimal if $m_1 = 0$.

In this case $\mathcal{A}$ is a deformation of the graded category $(\mathcal{A}, m_2)$.

Theorem (Kadeishvili'80, Lefèvre-Hasegawa'03)

Any A_∞-category $\mathcal{A}$ over a field k is quasi-isomorphic to a minimal one, defined over $H^*\mathcal{A}$.

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The Hochschild complex $C^{*,*}(\mathcal{A})$ *on a graded category* $\mathcal{A}$ *is*

$$C^{n,r}(\mathcal{A}) = \bigoplus_{X_0,\dots,X_n \text{ in } \mathcal{A}} \operatorname{Hom}^r(\mathcal{A}(X_{n-1},X_n)\otimes\cdots\otimes\mathcal{A}(X_0,X_1),\mathcal{A}(X_0,X_n)).$$

with differential ∂ *of degree* $(1,0)$.

The shifted Hochschild complex $C^{*+1,*}(\mathcal{A})$ is a DGLA with the Gerstenhaber bracket.

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A minimal A_∞-structure on a graded category $\mathcal{A}$ is a Hochschild cochain of total degree 2

$$m = m_3 + m_4 + \cdots + m_n + \cdots$$

concentrated in horizontal degrees ≥ 3 which is a solution of the Maurer–Cartan equation,

$$\partial(m) + \frac{1}{2}[m, m] = 0.$$

This equation can be decomposed as

$$\partial(m_n) + \frac{1}{2} \sum_{p+q=n+2} [m_p, m_q] = 0, \quad n \geq 3,$$

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An A_3-structure on a graded category is just a 3-cocycle m_3, $\{m_3\} \in HH^{3,-1}(\mathcal{A})$.

An A_∞-structure is a sequence of cochains $m_3, \dots, m_n, \dots$ such that $m_3, \dots, m_n$ is an A_n-structure for all $n \geq 3$.

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The derived category $D(\mathcal{A})$ of an A_∞-category $\mathcal{A}$ is the homotopy category of right $\mathcal{A}$-modules, which is triangulated in a natural way. The inclusion of free modules induces a functor

$$\begin{array}{rcl} H^0\mathcal{A} & \longrightarrow & D(\mathcal{A}) \\ X & \mapsto & \mathcal{A}(\,\cdot\,, X). \end{array}$$

Definition

An A_∞-category $\mathcal{A}$ is pretriangulated if $H^0\mathcal{A}$ is a triangulated subcategory of $D(\mathcal{A})$.

If $\mathcal{A}$ is pretriangulated and $\mathcal{T} = H^0(\mathcal{A})$ then

$$H^n\mathcal{A}(X, Y) \cong \mathcal{T}(X, \Sigma^n Y), \quad n \in \mathbb{Z},$$

where Σ is the suspension in $\mathcal{T}$.

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The problem

Let $\mathcal{T}$ be a triangulated category over k with suspension Σ.

When is $\mathcal{T}$ algebraic?

Translation: For k a field, we wonder about the existence of a minimal pretriangulated A_∞-category $\mathcal{A} = (\mathcal{T}_\Sigma, m)$ on the graded category $\mathcal{T}_\Sigma$ with the same objects as $\mathcal{T}$ and morphisms

$$\mathcal{T}_\Sigma(X, Y) = \bigoplus_{n \in \mathbb{Z}} \mathcal{T}(X, \Sigma^n Y),$$

such that $\mathcal{T}$ embeds as a triangulated subcategory of $D(\mathcal{A})$. For this we have to find $m_3, m_4, \ldots$ adequately.

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Secondary compositions

A secondary composition or Massey product or Toda bracket in an additive graded category $\mathcal{C}$ is an operation which sends composable homogeneous morphisms

$$Z \xrightarrow{h} Y \xrightarrow{g} X \xrightarrow{f} W,$$

with $fg = 0$ and $gh = 0$, to

$$\langle f, g, h \rangle \quad \in \quad \frac{\mathcal{C}(Z, W)}{f \cdot \mathcal{C}(Z, X) + \mathcal{C}(Y, W) \cdot h},$$

such that

$$\deg(\langle f, g, h \rangle) \quad = \quad \deg(f) + \deg(g) + \deg(h) - 1,$$

$$\langle f, g, h \rangle \cdot i \subset \langle f, g, h \cdot i \rangle \subset \langle f, g \cdot h, i \rangle \supset \langle f \cdot g, h, i \rangle \supset (-1)^{\deg(f)} f \cdot \langle g, h, i \rangle.$$

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Secondary compositions in $\mathcal{T}_\Sigma$

The graded category $\mathcal{T}_\Sigma$ carries a secondary composition induced by the triangulated structure on $\mathcal{T}$. Given

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exact

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Secondary compositions in $\mathcal{T}_\Sigma$

Conversely, this secondary composition determines the exact triangles.

Proposition

A triangle $X \stackrel{f}{\to} Y \stackrel{i}{\to} C \stackrel{q}{\to} \Sigma X$ *is exact in* $\mathcal{T}$ *if and only if*

$$\mathcal{T}(U, X) \to \mathcal{T}(U, Y) \to \mathcal{T}(U, C) \to \mathcal{T}(U, \Sigma X) \to \mathcal{T}(U, \Sigma Y)$$

is exact for any object U *in* $\mathcal{T}$ *and* $1_X \in \langle q, i, f\rangle \subset \mathcal{T}(X, X)$.

Using [Heller'68] one can actually determine the subset

$$\{\text{Puppe triangulated structures in } \mathcal{T}\} \subseteq \{\text{Secondary compositions in } \mathcal{T}_\Sigma\}$$

which is the intersection of an 'open' and a 'closed' subset.

Secondary compositions in $\mathcal{T}_\Sigma$

Conversely, this secondary composition determines the exact triangles.

Proposition

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Secondary compositions in $\mathcal{T}_\Sigma$

Definition

A finitely presented right $\mathcal{T}$-module *is a functor* $M\colon \mathcal{T}^{\mathrm{op}} \to k\text{-Mod}$ *which fits into an exact sequence*

$$\mathcal{T}(\cdot, X) \to \mathcal{T}(\cdot, Y) \to M \to 0$$

Theorem (Freyd'66)

The category mod-$\mathcal{T}$ *of finitely presented right* $\mathcal{T}$*-modules is a Frobenius abelian category.*

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The suspension functor in $\mathcal{T}$ extends uniquely to an exact equivalence

$$\Sigma\colon \text{mod-}\mathcal{T} \longrightarrow \text{mod-}\mathcal{T}.$$

We can therefore define a graded category mod-$\mathcal{T}_\Sigma$ with the same objects as mod-$\mathcal{T}$ and graded morphisms

$$\mathrm{Hom}^*_{\mathcal{T}}(M,N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(M,\Sigma^n N),$$

and also bigraded ext's $\mathrm{Ext}^{*,*}_{\mathcal{T}}$.

Proposition

$\{$*Secondary compositions in* $\mathcal{T}_\Sigma\} \cong HH^{0,-1}(\text{mod-}\mathcal{T}_\Sigma, \mathrm{Ext}^{3,*}_{\mathcal{T}})$.

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Idea of the proof.

Suppose we have a secondary composition $\langle \cdot, \cdot, \cdot \rangle$. We now define an element in $\kappa \in HH^{0,-1}(\text{mod-}\mathcal{T}_\Sigma, \mathrm{Ext}^{3,*}_{\mathcal{T}})$. Let M be in mod-$\mathcal{T}$,

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The first obstructions

Theorem

For k a field and any $r \in \mathbb{Z}$ there is a spectral sequence

$$E_2^{p,q} = HH^{p,r}(\text{mod-}\mathcal{T}_\Sigma, \operatorname{Ext}_{\mathcal{T}}^{q,*}) \Longrightarrow HH^{p+q,r}(\mathcal{T}_\Sigma).$$

If $\mathcal{T} = H^0\mathcal{A}$ for some pretriangulated minimal A_∞-category $\mathcal{A}$ then the edge homomorphism for $r = -1$ satisfies

$$\begin{array}{ccccc} HH^{3,-1}(\mathcal{T}_\Sigma) & \longrightarrow & HH^{0,-1}(\text{mod-}\mathcal{T}_\Sigma, \operatorname{Ext}_{\mathcal{T}}^{3,*}) & = & E_2^{0,3} \\ \{m_3\} & \mapsto & \langle\cdot,\cdot,\cdot\rangle. & & \end{array}$$

Corollary

If $\mathcal{T}$ is algebraic over a field k then the secondary composition $\langle\cdot,\cdot,\cdot\rangle$ in $\mathcal{T}_\Sigma$ is a permanent cycle in the previous spectral sequence.

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Conversely, if $\langle\cdot,\cdot,\cdot\rangle$ is a permanent cycle we can choose a $(3,-1)$-cocycle m_3 such that

$$\{m_3\} \mapsto \langle\cdot,\cdot,\cdot\rangle$$

through the edge homomorphism.

Such a choice yields an A_3-structure on $\mathcal{T}_\Sigma$ that we can try to extend to an A_∞-structure.

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Therefore the first obstructions for the existence of an A_∞-enhancement are

$$\begin{array}{rrcl} & d_2(\langle\cdot,\cdot,\cdot\rangle) & \in & E_2^{2,2} = HH^{2,-1}(\text{mod-}\mathcal{T}_\Sigma, \operatorname{Ext}^{2,*}_{\mathcal{T}}), \quad \text{if } = 0 \\ \text{then} & d_3(\langle\cdot,\cdot,\cdot\rangle) & \in & E_3^{3,1}, \quad \text{if } = 0 \\ \text{then} & d_4(\langle\cdot,\cdot,\cdot\rangle) & \in & E_4^{4,0} \twoheadleftarrow HH^{4,-1}(\text{mod-}\mathcal{T}_\Sigma, \operatorname{Hom}^*_{\mathcal{T}}), \quad \text{if } = 0 \end{array}$$

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Higher obstructions

Suppose that we have enhanced $\mathcal{T}_\Sigma$ to an A_{n-1}-category, $n > 3$, in a compatible way with the triangulated structure of $\mathcal{T}$. Then

$$\left\{\frac{1}{2}\sum_{p+q=n+2}[m_p, m_q]\right\} \in HH^{n+1,2-n}(\mathcal{T}_\Sigma).$$

If this cohomology class vanishes then any trivialising cochain m_n yields an extension to an A_n-category since

$$\partial(m_n) + \frac{1}{2}\sum_{p+q=n+2}[m_p, m_q] \quad = \quad 0,$$

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The first of these higher obstructions is as follows.

Example

For $n = 4$, if $(\mathcal{T}_\Sigma, m_3)$ is an A_3-category, the obstruction for the existence of an A_4-enhancement is obtained from $\{m_3\} \in HH^{3,-1}(\mathcal{T}_\Sigma)$,

$$\frac{1}{2}[\{m_3\}, \{m_3\}] \quad \in \quad HH^{5,-2}(\mathcal{T}_\Sigma).$$

dual numbers Tate Amiot skip

The example of dual numbers

Let $\mathcal{T}$ = finitely generated free modules over $k[\varepsilon]/(\varepsilon^2)$ and Σ is the identity on objects and such that $\Sigma(\varepsilon) = -\varepsilon$.

$$\{\text{Secondary compositions in } \mathcal{T}_\Sigma\} \quad \cong \quad HH^{0,-1}(\text{mod-}\mathcal{T}_\Sigma, \mathrm{Ext}^{3,*}_{\mathcal{T}}) \quad \cong \quad k.$$

Each $x \in k^\times$ corresponds to the secondary composition of an algebraic triangulated structure on $\mathcal{T}$ with exact triangle

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The edge homomorphism is

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Let $\{m_3\} = x\cdot\alpha + y\cdot\beta$. The obstruction to enhance $(\mathcal{T}_\Sigma, m_3)$ to an A_4-category is

$$\begin{array}{rcl} \frac{1}{2}[x\cdot\alpha+y\cdot\beta, x\cdot\alpha+y\cdot\beta] & = & xy[\alpha,\beta]+\frac{1}{2}y^2[\beta,\beta], \\ \in\ HH^{5,-2}(\mathcal{T}_\Sigma) & \cong & k\cdot[\alpha,\beta]\oplus k\cdot[\beta,\beta], \quad [\alpha,\alpha]=0, \end{array}$$

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Another application of the spectral sequence

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An open problem

There are relevant finiteness conditions on triangulated categories which may allow cohomological computations:

- Krull–Remak–Schmidt.
- Finitely many indecomposables.
- Finite-dimensional hom's.

Over an algebraically closed field k, [Amiot'06] has classified the underlying category of a wide class of triangulated categories satisfying these conditions. This class includes maximal d-Calabi–Yau's, $d \geq 2$.

It could be interesting to determine how many of them are algebraic for $k = \overline{\mathbb{Q}}$. This could eventually yield examples of exotic triangulated categories where 2 and all primes are invertible.

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Main ingredients

The special features of fields which allow the definition of an obstruction theory for the existence of an A_∞-enhancement of a triangulated category over a field k are:

- Kadeishvili's theorem: any A_∞-category is quasi-isomorphic to a minimal one.
- The spectral sequence: $HH^{p,r}(\text{mod-}\mathcal{T}_\Sigma, \operatorname{Ext}_{\mathcal{T}}^{q,*}) \Rightarrow HH^{p+q,r}(\mathcal{T}_\Sigma)$.

What happens when k is just a commutative ring?

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Arbitrary commutative ground ring k

There is a version of Kadeishvili's theorem over an arbitrary commutative ring.

Theorem (Sagave'08)

Any A_∞-algebra is quasi-isomorphic to a minimal derived A_∞-algebra.

This theorem may be extended to A_∞-categories.

Derived A_∞-algebras are related to Shukla cohomology (a.k.a. derived Hochschild cohomology) as A_∞-algebras are related to Hochschild cohomology. In particular any derived A_∞-algebra $\mathcal{A}$ yields a characteristic cohomology class

$$\gamma_{\mathcal{A}} \in SH^{3,-1}(H^*\mathcal{A}).$$

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There is an ungraded version of the spectral sequence for Shukla cohomology.

Theorem (Lowen–van den Bergh'04)

If $\mathcal{T}$ is a triangulated category over k then there is a spectral sequence $E_2^{p,q} = SH^p(\text{mod-}\mathcal{T}, \operatorname{Ext}^q_{\mathcal{T}}) \Rightarrow SH^{p+q}(\mathcal{T})$.

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Over a field k, ungraded Hochschild cohomology is related to the graded version as follows.

Proposition

There are exact triangles in $D(k)$ for all $r, q \in \mathbb{Z}$,

$$C^{*,r}(\mathcal{T}_\Sigma) \to C^*(\mathcal{T}, \mathrm{Hom}^r_{\mathcal{T}}) \xrightarrow{1+\Sigma_*^{-1}\Sigma^*} C^*(\mathcal{T}, \mathrm{Hom}^r_{\mathcal{T}}) \to C^{*,r}(\mathcal{T}_\Sigma)[1],$$

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Corollary

The graded Hochschild cohomology $HH^{*,-1}(\mathcal{T}_\Sigma)$ is translation Hochschild cohomology $HH^*(\mathcal{T}, \Sigma)$ [Baues–M.'07].

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A triangulated category $\mathcal{T}$ is topological if it is equivalent to a full triangulated subcategory of a stable homotopy category.

Theorem (Dugger'06,...)

If $\mathcal{T}$ is compactly generated then $\mathcal{T}$ is topological $\Leftrightarrow \mathcal{T} = \pi_0 \mathcal{S}$ for a pretriangulated spectral category $\mathcal{S}$.

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We do not know of any version of Kadeishvili's theorem for ring spectra.

The topological Hochschild cohomology of an additive category $\mathcal{C}$ is equivalent to the Baues–Wirsching cohomology of $\mathcal{C}$ [Pirashvili–Waldhausen'92, Dundas].

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If $\mathcal{T}$ is topological then any pretriangulated spectral category $\mathcal{S}$ with $\mathcal{T} = \pi_0\mathcal{S}$ yields a translation Baues–Wirsching cohomology class $\gamma_{\mathcal{S}} \in H^3(\mathcal{T}, \Sigma)$.

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Baues–Wirsching translation cohomology $H^*(\mathcal{T}, \Sigma)$ is defined as the cohomology of the complex $F(\mathcal{T}, \Sigma)$ fitting into the exact triangle

$$F(\mathcal{T}, \Sigma) \to THH(\mathcal{T}, \mathrm{Hom}_{\mathcal{T}}^{-1}) \xrightarrow{1+\Sigma_*^{-1}\Sigma^*} THH(\mathcal{T}, \mathrm{Hom}_{\mathcal{T}}^{-1}) \to F(\mathcal{T}, \Sigma)[1].$$

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Combining results of [Jibladze–Pirashvili'91] and [Ulmer'69] we obtain the following result.

Theorem

If $\mathcal{T}$ is a triangulated category then there is a spectral sequence $THH^p(\text{mod-}\mathcal{T}, \text{Ext}^q_{\mathcal{T}}) \Rightarrow THH^{p+q}(\mathcal{T})$.

The End

Thanks for your attention!