Finite triangulated categories and DG-enhancements

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Enhancements

Let k be a ground field, $\mathcal{T}$ triangulated category, and $\Sigma\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ its shift functor $\Sigma X = X[1]$, a.k.a. *suspension*.

An **enhancement** is a DG-category $\mathcal{A}$ such that

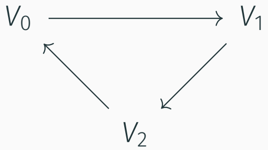
$$D^c(\mathcal{A}) \simeq \mathcal{T}.$$

A **Morita equivalence** is a DG-functor $\mathcal{A} \to \mathcal{B}$ which induces an equivalence $D^c(\mathcal{A}) \to D^c(\mathcal{B})$.

Problem: existence and uniqueness of enhancements up to Morita equivalence.

Let $\mathcal{T}$ be the category of finite-dimensional $\mathbb{C}$-vector spaces and $\Sigma = \mathsf{id}$.

Exact triangles are 3-periodic long exact sequences



An enhancement is $\mathbb{C}\langle t^{\pm 1}\rangle$ with $|t| = 1$.

Is it **unique?**

Any enhancement is Morita equivalent to a pre-triangulated one $\mathcal{A}$,

$$H^0(\mathcal{A}) \xrightarrow{\sim} D^c(\mathcal{A}) \simeq \mathcal{T}.$$

If the object C of $\mathcal{A}$ maps to $\mathbb{C}$ in $\mathcal{T}$, then $\mathcal{A}$ is Morita equivalent to the DG-algebra $A = \mathcal{A}(C, C)$, which must satisfy

$$H^*(A) \cong \mathcal{T}_\Sigma(\mathbb{C}, \mathbb{C}) = \mathbb{C}\langle t^{\pm 1}\rangle.$$

Here $\mathcal{T}_\Sigma$ is the graded category with the same objects as $\mathcal{T}$ and

$$\mathcal{T}_\Sigma(X, Y)^n = \mathcal{T}(X, \Sigma^n Y).$$

A toy example

An easy Hochschild cohomology computation yields

$$HH^{\star,*}(\mathbb{C}\langle t^{\pm 1}\rangle) = HH^{\star}(\mathbb{C})[t^{\pm 2}, \delta], \qquad |t| = (0,2), \quad |\delta| = (1,0).$$

This apparently vanishes for $\star \geq 3$, hence $\mathbb{C}\langle t^{\pm 1}\rangle$ is quasi-isomorphic to A, so we get uniqueness… *except that we didn't specify k!*

We definitely get uniqueness over $k = \mathbb{C}$ or $\mathbb{R}$.

What about $k = \mathbb{Q}$? UNKNOWN!

An example by [Rizzardo and Van den Bergh, 2019]

Let $\mathcal{T} = \mathsf{mod}(k) \times \mathsf{mod}(k)$ and let Σ be the twist.

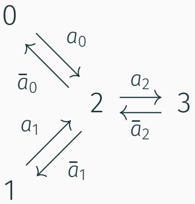
Exact triangles are 6-periodic long exact sequences,

$$
\begin{array}{ccc}
(V_0, V_3) & \longrightarrow & (V_1, V_4) \\
 & {\scriptstyle +1} \nwarrow \quad \swarrow & \\
 & (V_2, V_5) &
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & V_0 & \to & V_1 & \\
\nearrow & & & & \searrow \\
V_5 & & & & V_2 \\
\nwarrow & & & & \swarrow \\
 & V_4 & \leftarrow & V_3 &
\end{array}
$$

If $k = \ell(x_1, x_2, x_3)$ for ℓ a field of $\mathsf{char}\,\ell = 0$ then $\mathcal{T}$ has non-Morita equivalent enhancements over ℓ.

An example by [Amiot, 2007]

Let $\operatorname{char} k = 2$ and let $\mathcal{T} = \operatorname{proj}(\Lambda)$ be the category of finitely-generated projective right modules over the following deformed preprojective algebra Λ,



over an algebraically closed field of characteristic 2 by the two-sided ideal generated by

$$\bar{a}_0 a_0, \quad \bar{a}_1 a_1, \quad a_2 \bar{a}_2, \quad a_0 \bar{a}_0 + a_1 \bar{a}_1 + \bar{a}_2 a_2 + a_1 \bar{a}_1 a_0 \bar{a}_0,$$
$$a_0 \bar{a}_0 a_1 \bar{a}_1 + a_1 \bar{a}_1 a_0 \bar{a}_0.$$

An example by [Amiot, 2007]

Using a Λ-bimodule exact sequence by [Białkowski et al., 2007]

$$ {}_1\Lambda_{\sigma^{-1}} \hookrightarrow P_2 \to P_1 \to P_0 \twoheadrightarrow \Lambda, $$

with P_i projective, i.e. $\Omega^3_{\Lambda^{\mathrm{env}}}(\Lambda) = {}_1\Lambda_{\sigma^{-1}}$. Amiot defined a triangulated structure in $\mathcal{T}$ with Σ the restriction of scalars along $\sigma\colon \Lambda \cong \Lambda$ and exact triangles

$$ M \otimes_\Lambda (P_2 \to P_2 \to P_0 \to {}_1(P_2)_\sigma). $$

Here M runs over the finitely generated Λ-modules.

No enhancements were known.

Finite additive categories

An additive category $\mathcal{T}$ is **finite** if it is idempotent-complete, $\dim \mathcal{T}(X, Y) < \infty$ for any pair of objetcs, and there are finitely many indecomposables.

This is the same as having an equivalence $\mathcal{T} \simeq \text{proj}(\Lambda)$ for Λ a finite-dimensional basic algebra.

By [Freyd, 1966], if $\mathcal{T}$ is triangulated then Λ is self-injective.

Examples of finite triangulated categories

- $\underline{\text{mod}}(A)$ for A a finite-dimensional algebra of finite representation type.
- $\underline{\text{MCM}}(R)$ for R a commutative complete local ring of finite Cohen–Macaulay representation type.
- Orbit categories $D^b(kQ)/F^{\mathbb{Z}}$ for Q a Dynkin quiver and F an appropriate self-equivalence.
- The non-standard finite 1-Calabi–Yau categories of [Amiot, 2007], based on [Białkowski et al., 2007].

Structure theorems

For k algebraically closed.

- [Xiao and Zhu, 2005] computed the Auslander-Reiter quivers of finite triangulated categories.
- [Amiot, 2007] showed that in may cases $\mathcal{T}$ is equivalent to an orbit category as an additive category.
- [Keller, 2018] further showed that the previous equivalence is triangulated.

Previously known

Theorem [Hanihara, 2020]

Suppose k is a perfect field, Λ is a self-injective basic f.d. algebra, and $\mathcal{T} \simeq \mathsf{proj}(\Lambda)$. Then $\mathcal{T}$ has a Puppe triangulated structure if and only if $\Omega^3_{\Lambda^{\mathrm{env}}}(\Lambda)$ is stably isomorphic to an invertible Λ-bimodule.

Invertible Λ-bimodules are of the form ${}_1\Lambda_{\sigma^{-1}}$.

Warning! The given triangulated structure on $\mathcal{T}$ need not coincide with the one produced by Amiot's technique.

Main result, connected case

An **enhanced triangulated structure** on an additive category $\mathcal{T}$ consists of a DG-category $\mathcal{A}$ and an equivalence

$$D^c(\mathcal{A}) \simeq \mathcal{T}.$$

Theorem [Muro, 2020]

Suppose k is a perfect field, Λ is a connected non-separable self-injective basic f.d. algebra, and $\mathcal{T} \simeq \mathrm{proj}(\Lambda)$.Then $\mathcal{T}$ has an enhanced triangulated structure if and only if $\Omega^3_{\Lambda^{\mathrm{env}}}(\Lambda)$ is an invertible Λ-bimodule.In that case, $\Sigma^{-1} = - \otimes_\Lambda \Omega^3_{\Lambda^{\mathrm{env}}}(\Lambda)$. Moreover, the enhancement is unique up to Morita equivalence.

In the previous theorem, the underlying triangulated structure coincides with the one derived from Amiot's technique, so we obtain the following consequence.

Corollary

Amiot's non-standard finite 1-Calabi–Yau triangulated categories have enhancements.

Main result, general case

If $\mathcal{T}$ is an additive category and $\Sigma\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ is a self-equivalence, an **enhanced triangulated structure** on $(\mathcal{T}, \Sigma)$ consists of a DG-category $\mathcal{A}$ and an equivalence

$$D^c(\mathcal{A}) \simeq \mathcal{T}$$

compatible with the suspension functors.

Main result, general case

Theorem [Muro, 2020]

Suppose k is a perfect field, Λ is a self-injective basic f.d. algebra, and $\mathcal{T} \simeq \mathrm{proj}(\Lambda)$:

1. $\mathcal{T}$ has an enhanced triangulated structure if and only if $\Omega^3_{\Lambda^{\mathrm{env}}}(\Lambda)$ is stably isomorphic to an invertible Λ-bimodule.
2. The possible suspension functors $\Sigma\colon \mathcal{T} \to \mathcal{T}$ are $\Sigma \cong - \otimes_\Lambda I$ with $I^{-1} \cong \Omega^3_{\Lambda^{\mathrm{env}}}(\Lambda)$ in $\underline{\mathrm{mod}}(\Lambda^{\mathrm{env}})$.
3. If we fix Σ as above, any two enhanced triangulated structures on $(\mathcal{T}, \Sigma)$ are Morita equivalent.

Main result, general case

The following obvious consquence is connected to the example of [Rizzardo and Van den Bergh, 2019].

Corollary

If Λ is separable, then the enhanced triangulated structures on $\mathcal{T}$ are parametrized by $\mathrm{Out}(\Lambda)$.

Ideas from the proof

We consider the exact sequence of Λ-bimodules

$$I^{-1} = {}_1\Lambda_{\sigma^{-1}} \hookrightarrow P_2 \to P_1 \to P_0 \twoheadrightarrow \Lambda$$

with P_i projective as an element

$$\eta \in HH^3(\Lambda, I^{-1}).$$

If we define $\Lambda^\sigma = \bigoplus_{n\in\mathbb{Z}} I^{\otimes^n}$, its degree 0 part is Λ and

$$HH^3(\Lambda, I^{-1}) = HH^{3,-1}(\Lambda, \Lambda^\sigma).$$

Ideas from the proof

Since Λ is self-injective, we can consider Hochchild-Tate cohomology

$$\widehat{HH}^{\star,*}(\Lambda, \Lambda^{\sigma}).$$

Since the P_i are projective, η is a unit of degree $(3, -1)$.

We have morphisms

$$HH^{\star,*}(\Lambda^{\sigma}, \Lambda^{\sigma}) \longrightarrow HH^{\star,*}(\Lambda, \Lambda^{\sigma}) \longrightarrow \widehat{HH}^{\star,*}(\Lambda, \Lambda^{\sigma}).$$

The first one is induced by the inclusion $\Lambda \subset \Lambda^{\sigma}$ and the second one is the comparison map.

Ideas from the proof

Using that η is a unit in the target, we lift it to the source by a unique class $\{m_3\}$ satisfying

$$\frac{1}{2}[\{m_3\}, \{m_3\}] = 0$$

if $\operatorname{char} k \neq 2$ or more generally

$$\mathsf{Sq}(\{m_3\}) = 0.$$

We then show by means of an obstruction theory that the cocycle m_3 extends to a minimal A_∞-structure $(\Lambda^\sigma, m_3, m_4 \dots)$.

A DG-enhancement of this A_∞-algebra enhances $\mathcal{T}$.

Some questions

- How to obtain an explicit DG- or A_∞-enhancement?
- What happens in the *locally finite* case?
- What if k is not perfect?
- And if k is not even a field?
- Is there any finite triangulated category without enhancements, or even more, where the octahedral axiom fails?

That's all, thanks! ☺

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