On Massey products and triangulated categories

Fernando Muro

Universitat de Barcelona
Dept. Àlgebra i Geometria

Homotopy Theory and Higher Categories 2007–2008
Workshop on Derived Categories

Outline

- Massey products and Heller's theory.
- Cohomology of categories and Massey products.
- Stable Massey products and A_∞-enhancements.

Outline

- Massey products and Heller's theory.
- Cohomology of categories and Massey products.
- Stable Massey products and A_∞-enhancements.

Outline

- Massey products and Heller's theory.
- Cohomology of categories and Massey products.
- Stable Massey products and A_∞-enhancements.

Let k be a commutative ring.

Let $\mathcal{T}$ be a k-linear, additive, idempotent complete category.

A (right) $\mathcal{T}$-module M is a k-linear functor $M\colon \mathcal{T}^{\mathrm{op}} \to \text{Mod-}k$. It is finitely presented or coherent if there exists an exact sequence

$$\mathcal{T}(-,X) \longrightarrow \mathcal{T}(-,Y) \longrightarrow M \to 0.$$

Let $\text{mod-}\mathcal{T}$ be the category of coherent $\mathcal{T}$-modules.

Theorem (Freyd'66)

If $\mathcal{T}$ is triangulated then $\text{mod-}\mathcal{T}$ *is a Frobenius abelian category and $\mathcal{T}$ is the full subcategory of injective-projective objects.*

Assume that $\text{mod-}\mathcal{T}$ is a Frobenius ...

Let $\Sigma\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ be a k-linear equivalence, called suspension or translation.

Let k be a commutative ring.

Let $\mathcal{T}$ be a k-linear, additive, idempotent complete category.

A (right) $\mathcal{T}$-module M is a k-linear functor $M\colon \mathcal{T}^{\mathrm{op}} \to \text{Mod-}k$. It is finitely presented or coherent if there exists an exact sequence

$$\mathcal{T}(-,X) \longrightarrow \mathcal{T}(-,Y) \longrightarrow M \to 0.$$

Let $\text{mod-}\mathcal{T}$ be the category of coherent $\mathcal{T}$-modules.

Theorem (Freyd'66)

If $\mathcal{T}$ is triangulated then $\text{mod-}\mathcal{T}$ *is a Frobenius abelian category and $\mathcal{T}$ is the full subcategory of injective-projective objects.*

Assume that $\text{mod-}\mathcal{T}$ is a Frobenius ...

Let $\Sigma\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ be a k-linear equivalence, called suspension or translation.

Preliminaries

Let k be a commutative ring.

Let $\mathcal{T}$ be a k-linear, additive, idempotent complete category.

A (right) $\mathcal{T}$-module M is a k-linear functor $M\colon \mathcal{T}^{\mathrm{op}} \to \text{Mod-}k$. It is finitely presented or coherent if there exists an exact sequence

$$\mathcal{T}(-,X) \longrightarrow \mathcal{T}(-,Y) \longrightarrow M \to 0.$$

Let mod-$\mathcal{T}$ be the category of coherent $\mathcal{T}$-modules.

Theorem (Freyd'66)

If $\mathcal{T}$ is triangulated then mod-$\mathcal{T}$ *is a Frobenius abelian category and $\mathcal{T}$ is the full subcategory of injective-projective objects.*

Assume that mod-$\mathcal{T}$ is a Frobenius ...

Let $\Sigma\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ be a k-linear equivalence, called suspension or translation.

Let k be a commutative ring.

Let $\mathcal{T}$ be a k-linear, additive, idempotent complete category.

A (right) $\mathcal{T}$-module M is a k-linear functor $M\colon \mathcal{T}^{\mathrm{op}} \to \text{Mod-}k$. It is finitely presented or coherent if there exists an exact sequence

$$\mathcal{T}(-,X) \longrightarrow \mathcal{T}(-,Y) \longrightarrow M \to 0.$$

Let $\text{mod-}\mathcal{T}$ be the category of coherent $\mathcal{T}$-modules.

Theorem (Freyd'66)

If $\mathcal{T}$ is triangulated then $\text{mod-}\mathcal{T}$ *is a Frobenius abelian category and $\mathcal{T}$ is the full subcategory of injective-projective objects.*

Assume that $\text{mod-}\mathcal{T}$ is a Frobenius ...

Let $\Sigma\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ be a k-linear equivalence, called suspension or translation.

Let k be a commutative ring.

Let $\mathcal{T}$ be a k-linear, additive, idempotent complete category.

A (right) $\mathcal{T}$-module M is a k-linear functor $M\colon \mathcal{T}^{\mathrm{op}} \to \text{Mod-}k$. It is finitely presented or coherent if there exists an exact sequence

$$\mathcal{T}(-,X) \longrightarrow \mathcal{T}(-,Y) \longrightarrow M \to 0.$$

Let $\text{mod-}\mathcal{T}$ be the category of coherent $\mathcal{T}$-modules.

Theorem (Freyd'66)

If $\mathcal{T}$ is triangulated then $\text{mod-}\mathcal{T}$ is a Frobenius abelian category and $\mathcal{T}$ is the full subcategory of injective-projective objects.

Assume that $\text{mod-}\mathcal{T}$ is a Frobenius ...

Let $\Sigma\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ be a k-linear equivalence, called suspension or translation.

Preliminaries

Let k be a commutative ring.

Let $\mathcal{T}$ be a k-linear, additive, idempotent complete category.

A (right) $\mathcal{T}$-module M is a k-linear functor $M\colon \mathcal{T}^{\mathrm{op}} \to \text{Mod-}k$. It is finitely presented or coherent if there exists an exact sequence

$$\mathcal{T}(-,X) \longrightarrow \mathcal{T}(-,Y) \longrightarrow M \to 0.$$

Let $\text{mod-}\mathcal{T}$ be the category of coherent $\mathcal{T}$-modules.

Theorem (Freyd'66)

If $\mathcal{T}$ is triangulated then $\text{mod-}\mathcal{T}$ is a Frobenius abelian category and $\mathcal{T}$ is the full subcategory of injective-projective objects.

Assume that $\text{mod-}\mathcal{T}$ is a Frobenius ...

Let $\Sigma\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ be a k-linear equivalence, called suspension or translation.

Let k be a commutative ring.

Let $\mathcal{T}$ be a k-linear, additive, idempotent complete category.

A (right) $\mathcal{T}$-module M is a k-linear functor $M\colon \mathcal{T}^{\mathrm{op}} \to \operatorname{Mod-}k$. It is finitely presented or coherent if there exists an exact sequence

$$\mathcal{T}(-,X) \longrightarrow \mathcal{T}(-,Y) \longrightarrow M \to 0.$$

Let $\operatorname{mod-}\mathcal{T}$ be the category of coherent $\mathcal{T}$-modules.

Theorem (Freyd'66)

If $\mathcal{T}$ is triangulated then $\operatorname{mod-}\mathcal{T}$ is a Frobenius abelian category and $\mathcal{T}$ is the full subcategory of injective-projective objects.

Assume that $\operatorname{mod-}\mathcal{T}$ is a Frobenius ...

Let $\Sigma\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ be a k-linear equivalence, called suspension or translation.

A Massey product or secondary composition sends

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, dashed, "0"] & Y \arrow[r, "g"] \arrow[rr, bend right, dashed, "0"'] & Z \arrow[r, "h"] & U \end{tikzcd} \qquad \text{in } \mathcal{T}$$

to

$$\langle h, g, f\rangle \subset \mathcal{T}(\Sigma X, U),$$

a coset of

$$h \cdot \mathcal{T}(\Sigma X, Z) + \mathcal{T}(\Sigma Y, U) \cdot (\Sigma f) \subset \mathcal{T}(\Sigma X, U),$$

the indeterminacy submodule.

Massey products

Moreover, given composable morphisms (without vanishing conditions)

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} U \xrightarrow{i} V,$$

the following inclusions hold whenever the Massey products are defined,

$$\langle i, h, g\rangle \cdot (\Sigma f) \subset \langle i, h, g \cdot f\rangle \subset \langle i, h \cdot g, f\rangle \supset \langle i \cdot h, g, f\rangle \supset i \cdot \langle h, g, f\rangle.$$

The set of Massey products is a k-module,

$$\mathrm{MP}(\mathcal{T}, \Sigma).$$

Theorem (Heller'68)

If $\mathcal{T}$ is triangulated there is a unique Massey product such that for any exact triangle

$$X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{q} \Sigma X$$

we have

$$1_{\Sigma X} \in \langle q, i, f \rangle \subset \mathcal{T}(\Sigma X, \Sigma X).$$

This defines an inclusion

$$\{\textit{triangulated structures on } (\mathcal{T}, \Sigma)\} \quad \subset \quad \mathrm{MP}(\mathcal{T}, \Sigma).$$

▸ skip proof

Theorem (Heller'68)

If $\mathcal{T}$ is triangulated there is a unique Massey product such that for any exact triangle

$$X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{q} \Sigma X$$

we have

$$1_{\Sigma X} \in \langle q, i, f \rangle \subset \mathcal{T}(\Sigma X, \Sigma X).$$

This defines an inclusion

$$\{\textit{triangulated structures on } (\mathcal{T}, \Sigma)\} \quad \subset \quad \mathrm{MP}(\mathcal{T}, \Sigma).$$

skip proof

Idea of the proof.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, dashed, "0"] & Y \arrow[r, "g"] \arrow[rr, bend right, dashed, "0"'] & Z \arrow[r, "h"] & U \end{tikzcd}$$

exact

A triangle $X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{q} \Sigma X$ is exact if and only if

$$\mathcal{T}(-,X) \xrightarrow{f_*} \mathcal{T}(-,Y) \xrightarrow{i_*} \mathcal{T}(-,C) \xrightarrow{q_*} \mathcal{T}(-,\Sigma X) \xrightarrow{(\Sigma f)_*} \mathcal{T}(-,\Sigma Y)$$

is an exact sequence of $\mathcal{T}$-modules and $1_{\Sigma X} \in \langle q, i, f \rangle$. $\square$

Idea of the proof.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] & U \\ X \arrow[r, "f"'] & Y \arrow[r, "i"'] & C \arrow[r, "q"'] & \Sigma X \end{tikzcd}$$

exact

A triangle $X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{q} \Sigma X$ is exact if and only if

$$\mathcal{T}(-,X) \xrightarrow{f_*} \mathcal{T}(-,Y) \xrightarrow{i_*} \mathcal{T}(-,C) \xrightarrow{q_*} \mathcal{T}(-,\Sigma X) \xrightarrow{(\Sigma f)_*} \mathcal{T}(-,\Sigma Y)$$

is an exact sequence of $\mathcal{T}$-modules and $1_{\Sigma X} \in \langle q, i, f \rangle$. □

Idea of the proof.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] \arrow[rr, bend left, dashed, "0"] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] & U \\ X \arrow[r, "f"'] & Y \arrow[r, "i"'] & C \arrow[r, "q"'] & \Sigma X \end{tikzcd}$$

exact

A triangle $X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{q} \Sigma X$ is exact if and only if

$$\mathcal{T}(-,X) \xrightarrow{f_*} \mathcal{T}(-,Y) \xrightarrow{i_*} \mathcal{T}(-,C) \xrightarrow{q_*} \mathcal{T}(-,\Sigma X) \xrightarrow{(\Sigma f)_*} \mathcal{T}(-,\Sigma Y)$$

is an exact sequence of $\mathcal{T}$-modules and $1_{\Sigma X} \in \langle q, i, f \rangle$. □

Idea of the proof.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] \arrow[rr, bend left, dashed, "0"] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] & U \\
X \arrow[r, "f"'] & Y \arrow[r, "i"'] & C \arrow[r, "q"'] \arrow[u, "a"'] & \Sigma X
\end{tikzcd}$$

exact

A triangle $X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{q} \Sigma X$ is exact if and only if

$$\mathcal{T}(-,X) \xrightarrow{f_*} \mathcal{T}(-,Y) \xrightarrow{i_*} \mathcal{T}(-,C) \xrightarrow{q_*} \mathcal{T}(-,\Sigma X) \xrightarrow{(\Sigma f)_*} \mathcal{T}(-,\Sigma Y)$$

is an exact sequence of $\mathcal{T}$-modules and $1_{\Sigma X} \in \langle q, i, f \rangle$. $\square$

Idea of the proof.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] \arrow[rr, bend left, dashed, "0"] & Z \arrow[r, "h"] & U \\
X \arrow[r, "f"'] & Y \arrow[r, "i"'] & C \arrow[r, "q"'] \arrow[u, "a"'] & \Sigma X
\end{tikzcd} \qquad \text{exact}$$

A triangle $X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{q} \Sigma X$ is exact if and only if

$$\mathcal{T}(-,X) \xrightarrow{f_*} \mathcal{T}(-,Y) \xrightarrow{i_*} \mathcal{T}(-,C) \xrightarrow{q_*} \mathcal{T}(-,\Sigma X) \xrightarrow{(\Sigma f)_*} \mathcal{T}(-,\Sigma Y)$$

is an exact sequence of $\mathcal{T}$-modules and $1_{\Sigma X} \in \langle q, i, f \rangle$. $\square$

Idea of the proof.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] \arrow[rr, bend left, dashed, "0"] & Z \arrow[r, "h"] & U \\ X \arrow[r, "f"'] & Y \arrow[r, "i"'] & C \arrow[r, "q"'] \arrow[u, "a"'] & \Sigma X \arrow[u, "b"'] \end{tikzcd}$$ exact

A triangle $X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{q} \Sigma X$ is exact if and only if

$$\mathcal{T}(-,X) \xrightarrow{f_*} \mathcal{T}(-,Y) \xrightarrow{i_*} \mathcal{T}(-,C) \xrightarrow{q_*} \mathcal{T}(-,\Sigma X) \xrightarrow{(\Sigma f)_*} \mathcal{T}(-,\Sigma Y)$$

is an exact sequence of $\mathcal{T}$-modules and $1_{\Sigma X} \in \langle q, i, f \rangle$. □

Idea of the proof.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & U \\ X \arrow[u, equal] \arrow[r, "f"'] & Y \arrow[u, equal] \arrow[r, "i"'] & C \arrow[u, "a"'] \arrow[r, "q"'] & \Sigma X \arrow[u, "b"'] \end{tikzcd} \quad b \in \langle h,g,f\rangle$$

exact

A triangle $X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{q} \Sigma X$ is exact if and only if

$$\mathcal{T}(-,X) \xrightarrow{f_*} \mathcal{T}(-,Y) \xrightarrow{i_*} \mathcal{T}(-,C) \xrightarrow{q_*} \mathcal{T}(-,\Sigma X) \xrightarrow{(\Sigma f)_*} \mathcal{T}(-,\Sigma Y)$$

is an exact sequence of $\mathcal{T}$-modules and $1_{\Sigma X} \in \langle q,i,f\rangle$. □

Idea of the proof.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & U \\ X \arrow[u, equal] \arrow[r, "f"'] & Y \arrow[u, equal] \arrow[r, "i"'] & C \arrow[u, "a"'] \arrow[r, "q"'] & \Sigma X \arrow[u, "b \in \langle h,g,f \rangle"'] \end{tikzcd}$$

exact

A triangle $X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{q} \Sigma X$ is exact if and only if

$$\mathcal{T}(-,X) \xrightarrow{f_*} \mathcal{T}(-,Y) \xrightarrow{i_*} \mathcal{T}(-,C) \xrightarrow{q_*} \mathcal{T}(-,\Sigma X) \xrightarrow{(\Sigma f)_*} \mathcal{T}(-,\Sigma Y)$$

is an exact sequence of $\mathcal{T}$-modules and $1_{\Sigma X} \in \langle q, i, f \rangle$. □

When is a Massey product induced by a triangulated structure?

Let $\underline{\mathrm{mod}}\text{-}\mathcal{T}$ be the stable category of coherent $\mathcal{T}$-modules,

$$\underline{\mathrm{Hom}}_{\mathcal{T}}(M,N) = \frac{\mathrm{Hom}_{\mathcal{T}}(M,N)}{\{M \to \mathcal{T}(-,X) \to N\}}.$$

The stable category is triangulated. The translation functor

$$S\colon \underline{\mathrm{mod}}\text{-}\mathcal{T} \longrightarrow \underline{\mathrm{mod}}\text{-}\mathcal{T}$$

is determined by the choice of short exact sequences in mod-$\mathcal{T}$,

$$0 \to M \longrightarrow \mathcal{T}(-,CM) \longrightarrow SM \to 0.$$

When is a Massey product induced by a triangulated structure?

Let $\underline{\mathrm{mod}}\text{-}\mathcal{T}$ be the stable category of coherent $\mathcal{T}$-modules,

$$\underline{\mathrm{Hom}}_{\mathcal{T}}(M,N) = \frac{\mathrm{Hom}_{\mathcal{T}}(M,N)}{\{M \to \mathcal{T}(-,X) \to N\}}.$$

The stable category is triangulated. The translation functor

$$S\colon \underline{\mathrm{mod}}\text{-}\mathcal{T} \longrightarrow \underline{\mathrm{mod}}\text{-}\mathcal{T}$$

is determined by the choice of short exact sequences in $\mathrm{mod}\text{-}\mathcal{T}$,

$$0 \to M \longrightarrow \mathcal{T}(-,CM) \longrightarrow SM \to 0.$$

When is a Massey product induced by a triangulated structure?

Let $\underline{\mathrm{mod}}\text{-}\mathcal{T}$ be the stable category of coherent $\mathcal{T}$-modules,

$$\underline{\mathrm{Hom}}_{\mathcal{T}}(M,N) \quad = \quad \frac{\mathrm{Hom}_{\mathcal{T}}(M,N)}{\{M \to \mathcal{T}(-,X) \to N\}}.$$

The stable category is triangulated. The translation functor

$$S\colon \underline{\mathrm{mod}}\text{-}\mathcal{T} \longrightarrow \underline{\mathrm{mod}}\text{-}\mathcal{T}$$

is determined by the choice of short exact sequences in mod-$\mathcal{T}$,

$$0 \to M \longrightarrow \mathcal{T}(-,CM) \longrightarrow SM \to 0.$$

The functor Σ extends in an essentially unique way,

$$\begin{tikzcd}
\mathcal{T} \arrow[rr, "\Sigma", "\sim"'] \arrow[d, "\text{Yoneda}"'] & & \mathcal{T} \arrow[d, "\text{Yoneda}"] & \\
\text{mod-}\mathcal{T} \arrow[rr, "\Sigma", "\sim"'] \arrow[d, two heads] & & \text{mod-}\mathcal{T} \arrow[d, two heads] & \text{exact} \\
\underline{\text{mod}}\text{-}\mathcal{T} \arrow[rr, "\Sigma", "\sim"'] & & \underline{\text{mod}}\text{-}\mathcal{T} & \text{triangle}
\end{tikzcd}$$

Heller's theory

Theorem (Heller'68)

There is a bijective correspondence between Puppe triangulated structures on $(\mathcal{T}, \Sigma)$ and natural isomorphisms $\delta\colon \Sigma \cong S^3$ such that for any coherent $\mathcal{T}$-module M,

$$\begin{tikzcd} \Sigma SM \arrow[rr, "\delta_{SM}"] \arrow[dr, "\cong"'] & \boxed{-1} & S^4 M \\ & S\Sigma M \arrow[ur, "S\delta_M"'] & \end{tikzcd}$$

Theorem

There is an isomorphism which sends the Massey product of a triangulation on $(\mathcal{T}, \Sigma)$ to Heller's natural isomorphism,

$$\mathrm{MP}(\mathcal{T}, \Sigma) \quad \cong \quad \mathrm{Hom}(\Sigma, S^3).$$

skip proof

Theorem (Heller'68)

There is a bijective correspondence between Puppe triangulated structures on $(\mathcal{T}, \Sigma)$ and natural isomorphisms $\delta\colon \Sigma \cong S^3$ such that for any coherent $\mathcal{T}$-module M,

$$\begin{tikzcd} \Sigma S M \arrow[rr, "\delta_{SM}"] \arrow[dr, "\cong"'] & \boxed{-1} & S^4 M \\ & S\Sigma M \arrow[ur, "S\delta_M"'] & \end{tikzcd}$$

Theorem

There is an isomorphism which sends the Massey product of a triangulation on $(\mathcal{T}, \Sigma)$ to Heller's natural isomorphism,

$$\mathrm{MP}(\mathcal{T}, \Sigma) \quad \cong \quad \mathrm{Hom}(\Sigma, S^3).$$

skip proof

Idea of the proof.

Let $\langle -,-,-\rangle$ be a Massey product. We need to define a morphism $\delta_M\colon \Sigma M \to S^3M$ for any coherent $\mathcal{T}$-module M.

$$\begin{tikzcd}[column sep=small]
& \mathcal{T}(-,CM) \arrow[rr, "f"] \arrow[dr] & & \mathcal{T}(-,CSM) \arrow[rr, "g"] \arrow[dr] & & \mathcal{T}(-,CS^2M) \arrow[rr, "h"] \arrow[dr] & & \mathcal{T}(-,CS^3M) \\
M \arrow[ur, hook] & & SM \arrow[ur, hook] & & S^2M \arrow[ur, hook] & & S^3M \arrow[ur, hook] &
\end{tikzcd}$$

Heller's theory

Idea of the proof.

Let $\langle -,-,- \rangle$ be a Massey product. We need to define a morphism $\delta_M \colon \Sigma M \to S^3 M$ for any coherent $\mathcal{T}$-module M.

$$\begin{tikzcd}
 & \mathcal{T}(-,CM) \arrow[rr, "f"] \arrow[dr, two heads] & & \mathcal{T}(-,CSM) \arrow[rr, "g"] \arrow[dr, two heads] & & \mathcal{T}(-,CS^2M) \arrow[rr, "h"] \arrow[dr, two heads] & & \mathcal{T}(-,CS^3M) \\
M \arrow[ur, hook] & & SM \arrow[ur, hook] & & S^2M \arrow[ur, hook] & & S^3M \arrow[ur, hook] &
\end{tikzcd}$$

Idea of the proof.

Let $\langle -, -, - \rangle$ be a Massey product. We need to define a morphism $\delta_M \colon \Sigma M \to S^3 M$ for any coherent $\mathcal{T}$-module M.

$$
\begin{tikzcd}[column sep=small]
 & \mathcal{T}(-, CM) \arrow[rr, "f"] \arrow[dr, two heads] & & \mathcal{T}(-, CSM) \arrow[rr, "g"] \arrow[dr, two heads] & & \mathcal{T}(-, CS^2M) \arrow[rr, "h"] \arrow[dr, two heads] & & \mathcal{T}(-, CS^3M) \\
M \arrow[ur, hook] & & SM \arrow[ur, hook] & & S^2M \arrow[ur, hook] & & S^3M \arrow[ur, hook] &
\end{tikzcd}
$$

$\square$

Idea of the proof.

Let $\langle -,-,-\rangle$ be a Massey product. We need to define a morphism $\delta_M\colon \Sigma M \to S^3M$ for any coherent $\mathcal{T}$-module M.

$$\begin{tikzcd}
 & \mathcal{T}(-,CM) \arrow[r, "f"] \arrow[dr, two heads] & \mathcal{T}(-,CSM) \arrow[r, "g"] \arrow[dr, two heads] & \mathcal{T}(-,CS^2M) \arrow[r, "h"] \arrow[dr, two heads] & \mathcal{T}(-,CS^3M) & \\
M \arrow[ur, hook] & & SM \arrow[ul, hook] & S^2M \arrow[ul, hook] & S^3M \arrow[ul, hook] & \mathcal{T}(-,\Sigma CM) \arrow[ul, "\langle h,g,f\rangle"']
\end{tikzcd}$$

$\square$

Heller's theory

Idea of the proof.

Let $\langle -,-,-\rangle$ be a Massey product. We need to define a morphism $\delta_M\colon \Sigma M \to S^3 M$ for any coherent $\mathcal{T}$-module M.

$$
\begin{tikzcd}[column sep=small]
 & \mathcal{T}(-,CM) \arrow[rr, "f"] \arrow[dr, two heads] & & \mathcal{T}(-,CSM) \arrow[rr, "g"] \arrow[dr, two heads] & & \mathcal{T}(-,CS^2M) \arrow[rr, "h"] \arrow[dr, two heads] & & \mathcal{T}(-,CS^3M) & \\
M \arrow[ur, hook] & & SM \arrow[ur, hook] & & S^2M \arrow[ur, hook] & & S^3M \arrow[ur, hook] & & \mathcal{T}(-,\Sigma CM) \arrow[ul, "\langle h,g,f\rangle"'] \\
 & & & & & & & \Sigma M \arrow[ur, hook] &
\end{tikzcd}
$$

$\square$

Heller's theory

Idea of the proof.

Let $\langle -,-,-\rangle$ be a Massey product. We need to define a morphism $\delta_M\colon \Sigma M \to S^3M$ for any coherent $\mathcal{T}$-module M.

$$\begin{tikzcd}
& \mathcal{T}(-,CM) \arrow[rr, "f"] \arrow[dr, two heads] & & \mathcal{T}(-,CSM) \arrow[rr, "g"] \arrow[dr, two heads] & & \mathcal{T}(-,CS^2M) \arrow[rr, "h"] \arrow[dr, two heads] & & \mathcal{T}(-,CS^3M) & \\
M \arrow[ur, hook] & & SM \arrow[ur, hook] & & S^2M \arrow[ur, hook] & & S^3M \arrow[ur, hook] & & \mathcal{T}(-,\Sigma CM) \arrow[ul, "{\langle h,g,f\rangle}"'] \\
& & & & & & & \Sigma M \arrow[ul, "\delta_M"] \arrow[ur, hook] &
\end{tikzcd}$$

$\square$

An example of Heller's theory

Let $\mathcal{T} = \mathcal{F}(\mathbb{Z}/4)$ be the category of finitely generated free $\mathbb{Z}/4$-modules and $\Sigma = 1_{\mathcal{F}(\mathbb{Z}/4)}$ the identity functor.

In this case mod-$\mathcal{T}$ = mod-$\mathbb{Z}/4$, $\underline{\text{mod}}$-$\mathcal{T} = \mathcal{F}(\mathbb{Z}/2)$ and $S = 1_{\mathcal{F}(\mathbb{Z}/2)}$.

$$\mathrm{MP}(\mathcal{F}(\mathbb{Z}/4), 1_{\mathcal{F}(\mathbb{Z}/4)}) \cong \mathrm{Hom}(1_{\mathcal{F}(\mathbb{Z}/2)}, 1_{\mathcal{F}(\mathbb{Z}/2)}) \cong \mathbb{Z}/2.$$

Theorem (M.-Schwede-Strickland'07)

The non-trivial Massey product in $(\mathcal{F}(\mathbb{Z}/4), 1_{\mathcal{F}(\mathbb{Z}/4)})$ is induced by a Verdier triangulated structure where the triangle

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4$$

is exact.

An example of Heller's theory

Let $\mathcal{T} = \mathcal{F}(\mathbb{Z}/4)$ be the category of finitely generated free $\mathbb{Z}/4$-modules and $\Sigma = 1_{\mathcal{F}(\mathbb{Z}/4)}$ the identity functor.

In this case $\operatorname{mod}\text{-}\mathcal{T} = \operatorname{mod}\text{-}\mathbb{Z}/4$, $\underline{\operatorname{mod}}\text{-}\mathcal{T} = \mathcal{F}(\mathbb{Z}/2)$ and $S = 1_{\mathcal{F}(\mathbb{Z}/2)}$.

$$\mathrm{MP}(\mathcal{F}(\mathbb{Z}/4), 1_{\mathcal{F}(\mathbb{Z}/4)}) \;\cong\; \mathrm{Hom}(1_{\mathcal{F}(\mathbb{Z}/2)}, 1_{\mathcal{F}(\mathbb{Z}/2)}) \;\cong\; \mathbb{Z}/2.$$

Theorem (M.-Schwede-Strickland'07)

The non-trivial Massey product in $(\mathcal{F}(\mathbb{Z}/4), 1_{\mathcal{F}(\mathbb{Z}/4)})$ is induced by a Verdier triangulated structure where the triangle

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4$$

is exact.

An example of Heller's theory

Let $\mathcal{T} = \mathcal{F}(\mathbb{Z}/4)$ be the category of finitely generated free $\mathbb{Z}/4$-modules and $\Sigma = 1_{\mathcal{F}(\mathbb{Z}/4)}$ the identity functor.

In this case $\operatorname{mod-}\mathcal{T} = \operatorname{mod-}\mathbb{Z}/4$, $\underline{\operatorname{mod}}\text{-}\mathcal{T} = \mathcal{F}(\mathbb{Z}/2)$ and $S = 1_{\mathcal{F}(\mathbb{Z}/2)}$.

$$\mathrm{MP}(\mathcal{F}(\mathbb{Z}/4), 1_{\mathcal{F}(\mathbb{Z}/4)}) \cong \operatorname{Hom}(1_{\mathcal{F}(\mathbb{Z}/2)}, 1_{\mathcal{F}(\mathbb{Z}/2)}) \cong \mathbb{Z}/2.$$

Theorem (M.-Schwede-Strickland'07)

The non-trivial Massey product in $(\mathcal{F}(\mathbb{Z}/4), 1_{\mathcal{F}(\mathbb{Z}/4)})$ is induced by a Verdier triangulated structure where the triangle

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4$$

is exact.

Let $\mathcal{T} = \mathcal{F}(\mathbb{Z}/4)$ be the category of finitely generated free $\mathbb{Z}/4$-modules and $\Sigma = 1_{\mathcal{F}(\mathbb{Z}/4)}$ the identity functor.

In this case mod-$\mathcal{T}$ = mod-$\mathbb{Z}/4$, $\underline{\text{mod}}$-$\mathcal{T} = \mathcal{F}(\mathbb{Z}/2)$ and $S = 1_{\mathcal{F}(\mathbb{Z}/2)}$.

$$\mathrm{MP}(\mathcal{F}(\mathbb{Z}/4), 1_{\mathcal{F}(\mathbb{Z}/4)}) \cong \mathrm{Hom}(1_{\mathcal{F}(\mathbb{Z}/2)}, 1_{\mathcal{F}(\mathbb{Z}/2)}) \cong \mathbb{Z}/2.$$

Theorem (M.-Schwede-Strickland'07)

The non-trivial Massey product in $(\mathcal{F}(\mathbb{Z}/4), 1_{\mathcal{F}(\mathbb{Z}/4)})$ is induced by a Verdier triangulated structure where the triangle

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4$$

is exact.

Hochschild-Mitchell cohomology

A $\mathcal{T}$-bimodule is a $\mathcal{T} \otimes \mathcal{T}^{\mathrm{op}}$-module.

The bar complex $C_*(\mathcal{T})$ is the complex of $\mathcal{T}$-bimodules

$$C_*(\mathcal{T}) = \bigoplus_{X_0,\dots,X_n} \mathcal{T}(X_0,-) \otimes \cdots \otimes \mathcal{T}(X_i, X_{i-1}) \otimes \cdots \otimes \mathcal{T}(-, X_n),$$

with differential

$$\partial(\alpha_0 \otimes \cdots \otimes \alpha_{n+1}) = \sum_{i=0}^{n} (-1)^i \alpha_0 \otimes \cdots \otimes (\alpha_i \alpha_{i+1}) \otimes \cdots \otimes \alpha_{n+1}.$$

The Hochschild-Mitchell cohomology of $\mathcal{T}$ with coefficients in M,

$$HH^*(\mathcal{T}, M),$$

is the cohomology of

$$C^*(\mathcal{T}, M) = \mathrm{Hom}_{\mathcal{T}\text{-bimod}}(C_*(\mathcal{T}), M).$$

Hochschild-Mitchell cohomology

A $\mathcal{T}$-bimodule is a $\mathcal{T} \otimes \mathcal{T}^{\mathrm{op}}$-module.

The bar complex $C_*(\mathcal{T})$ is the complex of $\mathcal{T}$-bimodules

$$C_*(\mathcal{T}) \quad = \quad \bigoplus_{X_0,\dots,X_n} \mathcal{T}(X_0,-) \otimes \cdots \otimes \mathcal{T}(X_i, X_{i-1}) \otimes \cdots \otimes \mathcal{T}(-,X_n),$$

with differential

$$\partial(\alpha_0 \otimes \cdots \otimes \alpha_{n+1}) \quad = \quad \sum_{i=0}^{n} (-1)^i \alpha_0 \otimes \cdots \otimes (\alpha_i \alpha_{i+1}) \otimes \cdots \otimes \alpha_{n+1}.$$

The Hochschild-Mitchell cohomology of $\mathcal{T}$ with coefficients in M,

$$HH^*(\mathcal{T}, M),$$

is the cohomology of

$$C^*(\mathcal{T}, M) \quad = \quad \mathrm{Hom}_{\mathcal{T}\text{-bimod}}(C_*(\mathcal{T}), M).$$

Hochschild-Mitchell cohomology

A $\mathcal{T}$-bimodule is a $\mathcal{T} \otimes \mathcal{T}^{\mathrm{op}}$-module.

The bar complex $C_*(\mathcal{T})$ is the complex of $\mathcal{T}$-bimodules

$$C_*(\mathcal{T}) = \bigoplus_{X_0,\dots,X_n} \mathcal{T}(X_0,-) \otimes \cdots \otimes \mathcal{T}(X_i, X_{i-1}) \otimes \cdots \otimes \mathcal{T}(-, X_n),$$

with differential

$$\partial(\alpha_0 \otimes \cdots \otimes \alpha_{n+1}) = \sum_{i=0}^{n} (-1)^i \alpha_0 \otimes \cdots \otimes (\alpha_i \alpha_{i+1}) \otimes \cdots \otimes \alpha_{n+1}.$$

The Hochschild-Mitchell cohomology of $\mathcal{T}$ with coefficients in M,

$$HH^*(\mathcal{T}, M),$$

is the cohomology of

$$C^*(\mathcal{T}, M) = \mathrm{Hom}_{\mathcal{T}\text{-bimod}}(C_*(\mathcal{T}), M).$$

Example

$\mathcal{T} = \mathcal{T}(-,-)$ is a $\mathcal{T}$-bimodule, and we denote

$$HH^*(\mathcal{T}) \quad = \quad HH^*(\mathcal{T},\mathcal{T}).$$

More generally, for any $q \in \mathbb{Z}$ we consider

$$HH^{p,q}(\mathcal{T}) \quad = \quad HH^p(\mathcal{T},\mathcal{T}(-,\Sigma^q)) \quad = \quad HH^p(\mathcal{T},\mathcal{T}(\Sigma^{-q},-)).$$

We also consider the $(\text{mod-}\mathcal{T})$-bimodules

$$\mathrm{Ext}_{\mathcal{T}}^{q,r} \quad = \quad \mathrm{Ext}_{\mathcal{T}}^{q}(-,\Sigma^r) \quad \cong \quad \mathrm{Ext}_{\mathcal{T}}^{q}(\Sigma^{-r},-), \qquad q \geq 0, \quad r \in \mathbb{Z},$$

and the cohomology

$$HH^p(\text{mod-}\mathcal{T},\mathrm{Ext}_{\mathcal{T}}^{q,r}).$$

Example

$\mathcal{T} = \mathcal{T}(-,-)$ is a $\mathcal{T}$-bimodule, and we denote

$$HH^*(\mathcal{T}) \quad = \quad HH^*(\mathcal{T},\mathcal{T}).$$

More generally, for any $q \in \mathbb{Z}$ we consider

$$HH^{p,q}(\mathcal{T}) \quad = \quad HH^p(\mathcal{T},\mathcal{T}(-,\Sigma^q)) \quad = \quad HH^p(\mathcal{T},\mathcal{T}(\Sigma^{-q},-)).$$

We also consider the $(\text{mod-}\mathcal{T})$*-bimodules*

$$\text{Ext}^{q,r}_{\mathcal{T}} \quad = \quad \text{Ext}^q_{\mathcal{T}}(-,\Sigma^r) \quad \cong \quad \text{Ext}^q_{\mathcal{T}}(\Sigma^{-r},-), \quad q \geq 0, \quad r \in \mathbb{Z},$$

and the cohomology

$$HH^p(\text{mod-}\mathcal{T},\text{Ext}^{q,r}_{\mathcal{T}}).$$

Example

$\mathcal{T} = \mathcal{T}(-,-)$ is a $\mathcal{T}$-bimodule, and we denote

$$HH^*(\mathcal{T}) \quad = \quad HH^*(\mathcal{T},\mathcal{T}).$$

More generally, for any $q \in \mathbb{Z}$ we consider

$$HH^{p,q}(\mathcal{T}) \quad = \quad HH^p(\mathcal{T},\mathcal{T}(-,\Sigma^q)) \quad = \quad HH^p(\mathcal{T},\mathcal{T}(\Sigma^{-q},-)).$$

We also consider the $(\text{mod-}\mathcal{T})$*-bimodules*

$$\mathrm{Ext}^{q,r}_{\mathcal{T}} \quad = \quad \mathrm{Ext}^q_{\mathcal{T}}(-,\Sigma^r) \quad \cong \quad \mathrm{Ext}^q_{\mathcal{T}}(\Sigma^{-r},-), \qquad q \geq 0, \quad r \in \mathbb{Z},$$

and the cohomology

$$HH^p(\text{mod-}\mathcal{T}, \mathrm{Ext}^{q,r}_{\mathcal{T}}).$$

Baues-Wirsching cohomology

The Baues-Wirsching cohomology of $\mathcal{T}$ with coefficients in M,

$$H^*(\mathcal{T}, M),$$

is the cohomology of the 'group ring' k-category $k[\mathcal{T}]$ obtained by taking free k-modules on morphism pointed sets,

$$k[\mathcal{T}](X, Y) \quad = \quad \text{free } k\text{-module on } \mathcal{T}(X, Y).$$

The natural k-linear functor $k[\mathcal{T}] \to \mathcal{T}$ induces a homomorphism

$$HH^*(\mathcal{T}, M) \longrightarrow H^*(\mathcal{T}, M).$$

Example

We consider $H^{p,q}(\mathcal{T}) = H^p(\mathcal{T}, \mathcal{T}(\Sigma^{-q}, -))$ *and* $H^p(\text{mod-}\mathcal{T}, \mathrm{Ext}^{q,r}_{\mathcal{T}})$.

The Baues-Wirsching cohomology of $\mathcal{T}$ with coefficients in M,

$$H^*(\mathcal{T}, M),$$

is the cohomology of the 'group ring' k-category $k[\mathcal{T}]$ obtained by taking free k-modules on morphism pointed sets,

$$k[\mathcal{T}](X, Y) \quad = \quad \text{free } k\text{-module on } \mathcal{T}(X, Y).$$

The natural k-linear functor $k[\mathcal{T}] \to \mathcal{T}$ induces a homomorphism

$$HH^*(\mathcal{T}, M) \longrightarrow H^*(\mathcal{T}, M).$$

Example

We consider $H^{p,q}(\mathcal{T}) = H^p(\mathcal{T}, \mathcal{T}(\Sigma^{-q}, -))$ *and* $H^p(\text{mod-}\mathcal{T}, \mathrm{Ext}^{q,r}_{\mathcal{T}})$.

The Baues-Wirsching cohomology of $\mathcal{T}$ with coefficients in M,

$$H^*(\mathcal{T}, M),$$

is the cohomology of the 'group ring' k-category $k[\mathcal{T}]$ obtained by taking free k-modules on morphism pointed sets,

$$k[\mathcal{T}](X, Y) \quad = \quad \text{free } k\text{-module on } \mathcal{T}(X, Y).$$

The natural k-linear functor $k[\mathcal{T}] \to \mathcal{T}$ induces a homomorphism

$$HH^*(\mathcal{T}, M) \longrightarrow H^*(\mathcal{T}, M).$$

Example

We consider $H^{p,q}(\mathcal{T}) = H^p(\mathcal{T}, \mathcal{T}(\Sigma^{-q}, -))$ *and* $H^p(\text{mod-}\mathcal{T}, \mathrm{Ext}^{q,r}_{\mathcal{T}})$.

A Baues-Wirsching $(3,-1)$-cocycle $z_{3,-1}$ of $\mathcal{T}$ sends any three composable morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} U$$

to an element

$$z_{3,-1}(h,g,f) \in \mathcal{T}(\Sigma X, U),$$

in such a way that

$$\begin{aligned} i \cdot z_{3,-1}(h,g,f) - z_{3,-1}(i \cdot h, g, f) + z_{3,-1}(i, h \cdot g, f) & \\ -z_{3,-1}(i,h,g \cdot f) + z_{3,-1}(i,h,g) \cdot (\Sigma f) &= 0. \end{aligned}$$

It is a Hochschild-Mitchell cocycle if $z_{3,-1}$ is k-multilinear.

Massey products and H^3

A Baues-Wirsching $(3,-1)$-cocycle $z_{3,-1}$ of $\mathcal{T}$ sends any three composable morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} U$$

to an element

$$z_{3,-1}(h,g,f) \in \mathcal{T}(\Sigma X, U),$$

in such a way that

$$\begin{aligned} i\cdot z_{3,-1}(h,g,f) - z_{3,-1}(i\cdot h,g,f) + z_{3,-1}(i,h\cdot g,f) & \\ -z_{3,-1}(i,h,g\cdot f) + z_{3,-1}(i,h,g)\cdot(\Sigma f) &= 0. \end{aligned}$$

It is a Hochschild-Mitchell cocycle if $z_{3,-1}$ is k-multilinear.

Lemma

Given a Baues-Wirsching $(3,-1)$-cocycle $z_{3,-1}$ there is defined a unique Massey product in $(\mathcal{T},\Sigma)$ such that

$$z_{3,-1}(h,g,f) \in \langle h,g,f\rangle \subset \mathcal{T}(\Sigma X, U).$$

This defines a homomorphism

$$HH^{3,-1}(\mathcal{T}) \longrightarrow H^{3,-1}(\mathcal{T}) \longrightarrow \mathrm{MP}(\mathcal{T},\Sigma).$$

Massey products and H^3

Lemma

Given a Baues-Wirsching $(3,-1)$-cocycle $z_{3,-1}$ there is defined a unique Massey product in $(\mathcal{T},\Sigma)$ such that

$$z_{3,-1}(h,g,f)\in\langle h,g,f\rangle\subset\mathcal{T}(\Sigma X,U).$$

This defines a homomorphism

$$HH^{3,-1}(\mathcal{T})\longrightarrow H^{3,-1}(\mathcal{T})\longrightarrow \mathrm{MP}(\mathcal{T},\Sigma).$$

Theorem (Pirashvili'88, Baues-Dreckmann'89)

The Massey product of a topological triangulated category is in the image of

$$H^{3,-1}(\mathcal{T}) \longrightarrow \mathrm{MP}(\mathcal{T},\Sigma).$$

The Massey product of a locally projective algebraic triangulated category is in the image of

$$HH^{3,-1}(\mathcal{T}) \longrightarrow \mathrm{MP}(\mathcal{T},\Sigma).$$

skip proof

Is there any triangulated category whose Massey product does not come from $HH^{3,-1}$ or $H^{3,-1}$?

Theorem (Pirashvili'88, Baues-Dreckmann'89)

The Massey product of a topological triangulated category is in the image of

$$H^{3,-1}(\mathcal{T}) \longrightarrow \mathrm{MP}(\mathcal{T},\Sigma).$$

The Massey product of a locally projective algebraic triangulated category is in the image of

$$HH^{3,-1}(\mathcal{T}) \longrightarrow \mathrm{MP}(\mathcal{T},\Sigma).$$

skip proof

Is there any triangulated category whose Massey product does not come from $HH^{3,-1}$ or $H^{3,-1}$?

Theorem (Pirashvili'88, Baues-Dreckmann'89)

The Massey product of a topological triangulated category is in the image of

$$H^{3,-1}(\mathcal{T}) \longrightarrow \mathrm{MP}(\mathcal{T},\Sigma).$$

The Massey product of a locally projective algebraic triangulated category is in the image of

$$HH^{3,-1}(\mathcal{T}) \longrightarrow \mathrm{MP}(\mathcal{T},\Sigma).$$

skip proof

Is there any triangulated category whose Massey product does not come from $HH^{3,-1}$ or $H^{3,-1}$?

Massey products and H^3

Idea of the proof.

Let $\mathcal{M}$ be a topological or algebraic model of $\mathcal{T}$ such that $\mathcal{T} \subset D(\mathcal{M})$ as a full triangulated subcategory. There is defined a derived 2-category $D_2(\mathcal{M})$, and a projection

$$\begin{tikzcd} D_2(\mathcal{M}) \arrow[r, two heads] & D(\mathcal{M}) \supset \mathcal{T}. \arrow[l, dashed, bend left] \end{tikzcd}$$

The obstruction to the existence of a splitting pseudofunctor is

$$\langle D_2(\mathcal{M})\rangle_{|\mathcal{T}} \in H^{3,-1}(\mathcal{T}) \quad \text{universal Massey product}$$

and maps to the Massey product of $\mathcal{T}$ by

$$H^{3,-1}(\mathcal{T}) \longrightarrow \mathrm{MP}(\mathcal{T}, \Sigma).$$

Idea of the proof.

Let $\mathcal{M}$ be a topological or algebraic model of $\mathcal{T}$ such that $\mathcal{T} \subset D(\mathcal{M})$ as a full triangulated subcategory. There is defined a derived 2-category $D_2(\mathcal{M})$, and a projection

$$D_2(\mathcal{M}) \twoheadrightarrow D(\mathcal{M}) \supset \mathcal{T}.$$

The obstruction to the existence of a splitting pseudofunctor is

$$\langle D_2(\mathcal{M})\rangle_{|\mathcal{T}} \in H^{3,-1}(\mathcal{T}) \quad \text{universal Massey product}$$

and maps to the Massey product of $\mathcal{T}$ by

$$H^{3,-1}(\mathcal{T}) \longrightarrow \mathrm{MP}(\mathcal{T}, \Sigma).$$

Massey products and H^3

Idea of the proof.

Let $\mathcal{M}$ be a topological or algebraic model of $\mathcal{T}$ such that $\mathcal{T} \subset D(\mathcal{M})$ as a full triangulated subcategory. There is defined a derived 2-category $D_2(\mathcal{M})$, and a projection

$$\begin{tikzcd} D_2(\mathcal{M}) \arrow[r, two heads] & D(\mathcal{M}) \supset \mathcal{T}. \arrow[l, bend left, dashed] \end{tikzcd}$$

The obstruction to the existence of a splitting pseudofunctor is

$$\langle D_2(\mathcal{M}) \rangle_{|\mathcal{T}} \in H^{3,-1}(\mathcal{T}) \quad \text{universal Massey product}$$

and maps to the Massey product of $\mathcal{T}$ by

$$H^{3,-1}(\mathcal{T}) \longrightarrow \mathrm{MP}(\mathcal{T}, \Sigma).$$

Idea of the proof.

Let $\mathcal{M}$ be a topological or algebraic model of $\mathcal{T}$ such that $\mathcal{T} \subset D(\mathcal{M})$ as a full triangulated subcategory. There is defined a derived 2-category $D_2(\mathcal{M})$, and a projection

$$D_2(\mathcal{M}) \twoheadrightarrow D(\mathcal{M}) \supset \mathcal{T}.$$

The obstruction to the existence of a splitting pseudofunctor is

$$\langle D_2(\mathcal{M})\rangle_{|\mathcal{T}} \in H^{3,-1}(\mathcal{T}) \quad \text{universal Massey product}$$

and maps to the Massey product of $\mathcal{T}$ by

$$H^{3,-1}(\mathcal{T}) \longrightarrow \mathrm{MP}(\mathcal{T}, \Sigma).$$

Massey products and H^3

Idea of the proof.

Let $\mathcal{M}$ be a topological or algebraic model of $\mathcal{T}$ such that $\mathcal{T} \subset D(\mathcal{M})$ as a full triangulated subcategory. There is defined a derived 2-category $D_2(\mathcal{M})$, and a projection

$$\begin{tikzcd} D_2(\mathcal{M}) \arrow[r, two heads] & D(\mathcal{M}) \supset \mathcal{T}. \arrow[l, dashed, bend left] \end{tikzcd}$$

The obstruction to the existence of a splitting pseudofunctor is

$$\langle D_2(\mathcal{M}) \rangle_{|\mathcal{T}} \in H^{3,-1}(\mathcal{T}) \quad \text{universal Massey product}$$

and maps to the Massey product of $\mathcal{T}$ by

$$H^{3,-1}(\mathcal{T}) \longrightarrow \mathrm{MP}(\mathcal{T}, \Sigma).$$

$\square$

Proposition

There is an isomorphism

$$\mathrm{MP}(\mathcal{T},\Sigma) \cong H^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}).$$

skip proof

Proof.

$$\begin{aligned}\mathrm{MP}(\mathcal{T},\Sigma) &\cong \mathrm{Hom}(\Sigma, S^3)\\ &\cong H^0(\underline{\mathrm{mod}}\text{-}\mathcal{T}, \underline{\mathrm{Hom}}_{\mathcal{T}}(\Sigma, S^3))\\ &\cong H^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}),\end{aligned}$$

since $\underline{\mathrm{Hom}}_{\mathcal{T}}(\Sigma M, S^3 N) \cong \mathrm{Ext}^3_{\mathcal{T}}(\Sigma M, N)$ and $\text{mod-}\mathcal{T} \twoheadrightarrow \underline{\mathrm{mod}}\text{-}\mathcal{T}$ is full and the identity on objects. $\square$

Proposition

There is an isomorphism

$$\mathrm{MP}(\mathcal{T}, \Sigma) \cong H^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}).$$

skip proof

Proof.

$$\begin{aligned}\mathrm{MP}(\mathcal{T}, \Sigma) &\cong \mathrm{Hom}(\Sigma, S^3)\\ &\cong H^0(\underline{\mathrm{mod}}\text{-}\mathcal{T}, \underline{\mathrm{Hom}}_{\mathcal{T}}(\Sigma, S^3))\\ &\cong H^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}),\end{aligned}$$

since $\underline{\mathrm{Hom}}_{\mathcal{T}}(\Sigma M, S^3 N) \cong \mathrm{Ext}^3_{\mathcal{T}}(\Sigma M, N)$ and $\text{mod-}\mathcal{T} \twoheadrightarrow \underline{\mathrm{mod}}\text{-}\mathcal{T}$ is full and the identity on objects. □

Massey products and H^0

Proposition

There is an isomorphism

$$\mathrm{MP}(\mathcal{T},\Sigma) \cong H^0(\mathrm{mod}\text{-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}).$$

skip proof

Proof.

$$\begin{aligned}\mathrm{MP}(\mathcal{T},\Sigma) &\cong \mathrm{Hom}(\Sigma, S^3)\\ &\cong H^0(\underline{\mathrm{mod}}\text{-}\mathcal{T}, \underline{\mathrm{Hom}}_{\mathcal{T}}(\Sigma, S^3))\\ &\cong H^0(\mathrm{mod}\text{-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}),\end{aligned}$$

since $\underline{\mathrm{Hom}}_{\mathcal{T}}(\Sigma M, S^3 N) \cong \mathrm{Ext}^3_{\mathcal{T}}(\Sigma M, N)$ and $\mathrm{mod}\text{-}\mathcal{T} \twoheadrightarrow \underline{\mathrm{mod}}\text{-}\mathcal{T}$ is full and the identity on objects. □

Proposition

There is an isomorphism

$$\mathrm{MP}(\mathcal{T}, \Sigma) \quad \cong \quad H^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}).$$

skip proof

Proof.

$$\begin{aligned}
\mathrm{MP}(\mathcal{T}, \Sigma) &\cong \mathrm{Hom}(\Sigma, S^3) \\
&\cong H^0(\underline{\mathrm{mod}}\text{-}\mathcal{T}, \underline{\mathrm{Hom}}_{\mathcal{T}}(\Sigma, S^3)) \\
&\cong H^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}),
\end{aligned}$$

since $\underline{\mathrm{Hom}}_{\mathcal{T}}(\Sigma M, S^3 N) \cong \mathrm{Ext}^3_{\mathcal{T}}(\Sigma M, N)$ and $\text{mod-}\mathcal{T} \twoheadrightarrow \underline{\mathrm{mod}}\text{-}\mathcal{T}$ is full and the identity on objects. $\square$

Theorem (Ulmer'69 + Jibladze-Pirashvili'91, Lowen-van den Bergh'05)

There is a spectral sequence for any $r \in \mathbb{Z}$,

$$H^p(\text{mod-}\mathcal{T}, \mathrm{Ext}^{q,r}_{\mathcal{T}}) \Longrightarrow H^{p+q,r}(\mathcal{T}),$$

and also for HH^* if for instance k is a field.

Proposition

The following diagram commutes (also for HH^* if k is a field).

$$\begin{tikzcd} & H^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}) \\ H^{3,-1}(\mathcal{T}) \arrow[ur, "edge"] \arrow[r, "\text{the previous one}"'] & \mathrm{MP}(\mathcal{T}, \Sigma) \arrow[u, "\cong"'] \end{tikzcd}$$

Massey products and H^0

Theorem (Ulmer'69 + Jibladze-Pirashvili'91, Lowen-van den Bergh'05)

There is a spectral sequence for any $r \in \mathbb{Z}$,

$$H^p(\text{mod-}\mathcal{T}, \mathrm{Ext}^{q,r}_{\mathcal{T}}) \Longrightarrow H^{p+q,r}(\mathcal{T}),$$

and also for HH^* if for instance k is a field.

Proposition

The following diagram commutes (also for HH^* if k is a field).

$$\begin{tikzcd} & H^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}) \\ H^{3,-1}(\mathcal{T}) \arrow[ur, "edge"] \arrow[r, "\text{the previous one}"'] & \mathrm{MP}(\mathcal{T}, \Sigma) \arrow[u, "\cong"'] \end{tikzcd}$$

Theorem (Ulmer'69 + Jibladze-Pirashvili'91, Lowen-van den Bergh'05)

There is a spectral sequence for any $r \in \mathbb{Z}$,

$$H^p(\text{mod-}\mathcal{T}, \operatorname{Ext}_{\mathcal{T}}^{q,r}) \Longrightarrow H^{p+q,r}(\mathcal{T}),$$

and also for HH^* if for instance k is a field.

Proposition

The following diagram commutes (also for HH^* if k is a field).

$$\begin{tikzcd} & H^0(\text{mod-}\mathcal{T}, \operatorname{Ext}_{\mathcal{T}}^{3,-1}) \\ H^{3,-1}(\mathcal{T}) \arrow[ur, "\text{edge}"] \arrow[r, "\text{the previous one}"'] & \operatorname{MP}(\mathcal{T}, \Sigma) \arrow[u, "\cong"'] \end{tikzcd}$$

The example $\mathcal{F}(\mathbb{Z}/4)$

Theorem

For $(\mathcal{F}(\mathbb{Z}/4), 1_{\mathcal{F}(\mathbb{Z}/4)})$ the edge homomorphism is trivial.

$$\mathbb{Z}/2 \cong HML^3(\mathbb{Z}/4) \cong H^{3,-1}(\mathcal{F}(\mathbb{Z}/4)) \xrightarrow{0} H^0(\text{mod-}\,\mathbb{Z}/4, \mathrm{Ext}^{3,-1}_{\mathbb{Z}/4}) \cong \mathbb{Z}/2.$$

Corollary (M.-Schwede-Strickland'07)

The triangulated category $\mathcal{F}(\mathbb{Z}/4)$ does not have any algebraic or topological model.

When does a triangulated category have a model? Is there an obstruction theory for the existence of models of any kind?

The example $\mathcal{F}(\mathbb{Z}/4)$

Theorem

For $(\mathcal{F}(\mathbb{Z}/4), 1_{\mathcal{F}(\mathbb{Z}/4)})$ the edge homomorphism is trivial.

$$\mathbb{Z}/2 \cong HML^3(\mathbb{Z}/4) \cong H^{3,-1}(\mathcal{F}(\mathbb{Z}/4)) \xrightarrow{0} H^0(\text{mod-}\mathbb{Z}/4, \mathrm{Ext}^{3,-1}_{\mathbb{Z}/4}) \cong \mathbb{Z}/2.$$

Corollary (M.-Schwede-Strickland'07)

The triangulated category $\mathcal{F}(\mathbb{Z}/4)$ does not have any algebraic or topological model.

When does a triangulated category have a model? Is there an obstruction theory for the existence of models of any kind?

The example $\mathcal{F}(\mathbb{Z}/4)$

Theorem

For $(\mathcal{F}(\mathbb{Z}/4), 1_{\mathcal{F}(\mathbb{Z}/4)})$ *the edge homomorphism is trivial.*

$$\mathbb{Z}/2 \cong HML^3(\mathbb{Z}/4) \cong H^{3,-1}(\mathcal{F}(\mathbb{Z}/4)) \xrightarrow{0} H^0(\text{mod-}\mathbb{Z}/4, \mathrm{Ext}^{3,-1}_{\mathbb{Z}/4}) \cong \mathbb{Z}/2.$$

Corollary (M.-Schwede-Strickland'07)

The triangulated category $\mathcal{F}(\mathbb{Z}/4)$ *does not have any algebraic or topological model.*

When does a triangulated category have a model? Is there an obstruction theory for the existence of models of any kind?

The example $\mathcal{F}(\mathbb{Z}/4)$

Theorem

For $(\mathcal{F}(\mathbb{Z}/4), 1_{\mathcal{F}(\mathbb{Z}/4)})$ *the edge homomorphism is trivial.*

$$\mathbb{Z}/2 \cong HML^3(\mathbb{Z}/4) \cong H^{3,-1}(\mathcal{F}(\mathbb{Z}/4)) \xrightarrow{0} H^0(\text{mod-}\,\mathbb{Z}/4, \mathrm{Ext}^{3,-1}_{\mathbb{Z}/4}) \cong \mathbb{Z}/2.$$

Corollary (M.-Schwede-Strickland'07)

The triangulated category $\mathcal{F}(\mathbb{Z}/4)$ *does not have any algebraic or topological model.*

When does a triangulated category have a model? Is there an obstruction theory for the existence of models of any kind?

The example $\mathcal{F}(\mathbb{Z}/4)$

Theorem

For $(\mathcal{F}(\mathbb{Z}/4), 1_{\mathcal{F}(\mathbb{Z}/4)})$ *the edge homomorphism is trivial.*

$$\mathbb{Z}/2 \cong HML^3(\mathbb{Z}/4) \cong H^{3,-1}(\mathcal{F}(\mathbb{Z}/4)) \xrightarrow{0} H^0(\text{mod-}\mathbb{Z}/4, \operatorname{Ext}^{3,-1}_{\mathbb{Z}/4}) \cong \mathbb{Z}/2.$$

Corollary (M.-Schwede-Strickland'07)

The triangulated category $\mathcal{F}(\mathbb{Z}/4)$ *does not have any algebraic or topological model.*

When does a triangulated category have a model? Is there an obstruction theory for the existence of models of any kind?

Stable Massey products

A Massey product on $(\mathcal{T}, \Sigma)$ is stable if

$$\langle \Sigma h, \Sigma g, \Sigma f\rangle \quad = \quad -\Sigma\langle h, g, f\rangle.$$

Therefore the submodule of stable Massey products $\mathrm{MP}_s(\mathcal{T}, \Sigma)$ is the kernel of

$$\mathrm{MP}(\mathcal{T}, \Sigma) \cong HH^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}) \xrightarrow{\Sigma_*^{-1}\Sigma^* + 1} HH^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}).$$

Moreover,

$$\{\text{triangulated structures on } (\mathcal{T}, \Sigma)\} \subset \mathrm{MP}_s(\mathcal{T}, \Sigma) \subset \mathrm{MP}(\mathcal{T}, \Sigma).$$

Stable Massey products

A Massey product on $(\mathcal{T},\Sigma)$ is stable if

$$\Sigma^{-1}\langle \Sigma h, \Sigma g, \Sigma f\rangle \quad = \quad -\langle h, g, f\rangle.$$

Therefore the submodule of stable Massey products $\mathrm{MP}_s(\mathcal{T},\Sigma)$ is the kernel of

$$\mathrm{MP}(\mathcal{T},\Sigma) \cong HH^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}) \xrightarrow{\Sigma_*^{-1}\Sigma^*+1} HH^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}).$$

Moreover,

$$\{\text{triangulated structures on } (\mathcal{T},\Sigma)\} \subset \mathrm{MP}_s(\mathcal{T},\Sigma) \subset \mathrm{MP}(\mathcal{T},\Sigma).$$

Stable Massey products

A Massey product on $(\mathcal{T}, \Sigma)$ is stable if

$$\Sigma^{-1}\langle \Sigma h, \Sigma g, \Sigma f\rangle \quad = \quad -\langle h, g, f\rangle.$$

Therefore the submodule of stable Massey products $\mathrm{MP}_s(\mathcal{T}, \Sigma)$ is the kernel of

$$\mathrm{MP}(\mathcal{T}, \Sigma) \cong HH^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}) \xrightarrow{\Sigma_*^{-1}\Sigma^*+1} HH^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}).$$

Moreover,

$$\{\text{triangulated structures on } (\mathcal{T}, \Sigma)\} \subset \mathrm{MP}_s(\mathcal{T}, \Sigma) \subset \mathrm{MP}(\mathcal{T}, \Sigma).$$

Stable Massey products

A Massey product on $(\mathcal{T}, \Sigma)$ is stable if

$$\Sigma^{-1}\langle \Sigma h, \Sigma g, \Sigma f\rangle \quad = \quad -\langle h, g, f\rangle.$$

Therefore the submodule of stable Massey products $\mathrm{MP}_s(\mathcal{T}, \Sigma)$ is the kernel of

$$\mathrm{MP}(\mathcal{T}, \Sigma) \cong HH^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}) \xrightarrow{\Sigma_*^{-1}\Sigma^*+1} HH^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}).$$

Moreover,

$$\{\text{triangulated structures on } (\mathcal{T}, \Sigma)\} \subset \mathrm{MP}_s(\mathcal{T}, \Sigma) \subset \mathrm{MP}(\mathcal{T}, \Sigma).$$

Cohomology of graded categories

Let k be a field and $\mathcal{T}_\Sigma$ the $\mathbb{Z}$-graded k-category with

$$\mathcal{T}_\Sigma(X,Y)^n = \mathcal{T}(X,\Sigma^n Y), \quad n \in \mathbb{Z}.$$

A $\mathcal{T}_\Sigma$-bimodule is a degree 0 functor $\mathcal{T}_\Sigma^{op} \otimes \mathcal{T}_\Sigma \to \mathrm{Mod}^{\mathbb{Z}}\text{-}\,k$ to $\mathbb{Z}$-graded k-modules.

The bar complex $C_*(\mathcal{T}_\Sigma)$ is now a complex of $\mathcal{T}_\Sigma$-bimodules.

Given a $\mathcal{T}_\Sigma$-bimodule M the Hochschild-Mitchell cohomology

$$HH^{p,q}(\mathcal{T}_\Sigma, M),$$

is the p^{th} cohomology of

$$C^*(\mathcal{T}_\Sigma, M[q]) = \mathrm{Hom}_{\mathcal{T}_\Sigma\text{-bimod}}(C_*(\mathcal{T}_\Sigma), M[q]).$$

Example

$\mathcal{T}_\Sigma = \mathcal{T}_\Sigma(-,-)$ is a $\mathcal{T}_\Sigma$-bimodule and $HH^{p,q}(\mathcal{T}_\Sigma) = HH^{p,q}(\mathcal{T}_\Sigma, \mathcal{T}_\Sigma)$.

Cohomology of graded categories

Let k be a field and $\mathcal{T}_\Sigma$ the $\mathbb{Z}$-graded k-category with

$$\mathcal{T}_\Sigma(X,Y)^n \quad = \quad \mathcal{T}(X,\Sigma^n Y), \qquad n\in\mathbb{Z}.$$

A $\mathcal{T}_\Sigma$-bimodule is a degree 0 functor $\mathcal{T}_\Sigma^{op}\otimes\mathcal{T}_\Sigma\rightarrow\operatorname{Mod}^{\mathbb{Z}}\text{-}k$ to $\mathbb{Z}$-graded k-modules.

The bar complex $C_*(\mathcal{T}_\Sigma)$ is now a complex of $\mathcal{T}_\Sigma$-bimodules.

Given a $\mathcal{T}_\Sigma$-bimodule M the Hochschild-Mitchell cohomology

$$HH^{p,q}(\mathcal{T}_\Sigma,M),$$

is the p^{th} cohomology of

$$C^*(\mathcal{T}_\Sigma,M[q]) \quad = \quad \operatorname{Hom}_{\mathcal{T}_\Sigma\text{-bimod}}(C_*(\mathcal{T}_\Sigma),M[q]).$$

Example

$\mathcal{T}_\Sigma=\mathcal{T}_\Sigma(-,-)$ is a $\mathcal{T}_\Sigma$-bimodule and $HH^{p,q}(\mathcal{T}_\Sigma)=HH^{p,q}(\mathcal{T}_\Sigma,\mathcal{T}_\Sigma)$.

Cohomology of graded categories

Let k be a field and $\mathcal{T}_\Sigma$ the $\mathbb{Z}$-graded k-category with

$$\mathcal{T}_\Sigma(X,Y)^n \quad = \quad \mathcal{T}(X,\Sigma^n Y), \quad n \in \mathbb{Z}.$$

A $\mathcal{T}_\Sigma$-bimodule is a degree 0 functor $\mathcal{T}_\Sigma^{op} \otimes \mathcal{T}_\Sigma \to \mathrm{Mod}^{\mathbb{Z}}\text{-}\, k$ to $\mathbb{Z}$-graded k-modules.

The bar complex $C_*(\mathcal{T}_\Sigma)$ is now a complex of $\mathcal{T}_\Sigma$-bimodules.

Given a $\mathcal{T}_\Sigma$-bimodule M the Hochschild-Mitchell cohomology

$$HH^{p,q}(\mathcal{T}_\Sigma, M),$$

is the p^{th} cohomology of

$$C^*(\mathcal{T}_\Sigma, M[q]) \quad = \quad \mathrm{Hom}_{\mathcal{T}_\Sigma\text{-bimod}}(C_*(\mathcal{T}_\Sigma), M[q]).$$

Example

$\mathcal{T}_\Sigma = \mathcal{T}_\Sigma(-,-)$ is a $\mathcal{T}_\Sigma$-bimodule and $HH^{p,q}(\mathcal{T}_\Sigma) = HH^{p,q}(\mathcal{T}_\Sigma, \mathcal{T}_\Sigma)$.

Cohomology of graded categories

Proposition

For any $q \in \mathbb{Z}$, the complex $C^*(\mathcal{T}_\Sigma, \mathcal{T}_\Sigma[q])$ is the homotopy fiber of

$$\Sigma_*^{-1}\Sigma^* + 1\colon C^*(\mathcal{T}, \mathcal{T}(-, \Sigma^q)) \longrightarrow C^*(\mathcal{T}, \mathcal{T}(-, \Sigma^q)).$$

This homotopy fiber is strongly related to the stability equation for Massey products,

$$\Sigma^{-1}\langle \Sigma h, \Sigma g, \Sigma f\rangle = -\langle h, g, f\rangle.$$

Cohomology of graded categories

Proposition

For any $q \in \mathbb{Z}$, the complex $C^*(\mathcal{T}_\Sigma, \mathcal{T}_\Sigma[q])$ is the homotopy fiber of

$$\Sigma_*^{-1}\Sigma^* + 1\colon C^*(\mathcal{T}, \mathcal{T}(-, \Sigma^q)) \longrightarrow C^*(\mathcal{T}, \mathcal{T}(-, \Sigma^q)).$$

This homotopy fiber is strongly related to the stability equation for Massey products,

$$\Sigma^{-1}\langle \Sigma h, \Sigma g, \Sigma f\rangle \quad = \quad -\langle h, g, f\rangle.$$

Corollary

There is a long exact sequence for any $q \in \mathbb{Z}$,

$$\cdots \to HH^{p,q}(\mathcal{T}_\Sigma) \to HH^{p,q}(\mathcal{T}) \xrightarrow{\Sigma_*^{-1}\Sigma^*+1} HH^{p,q}(\mathcal{T}) \to HH^{p+1,q}(\mathcal{T}_\Sigma) \to \cdots$$

Moreover, there is a commutative diagram

$$\begin{tikzcd} HH^{3,-1}(\mathcal{T}) \arrow[r, "edge"] & HH^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}) \cong \mathrm{MP}(\mathcal{T},\Sigma) \\ HH^{3,-1}(\mathcal{T}_\Sigma) \arrow[u] \arrow[r] & \mathrm{MP}_s(\mathcal{T},\Sigma) \arrow[u, hook] \end{tikzcd}$$

An element $\{m_3\} \in HH^{3,-1}(\mathcal{T}_\Sigma)$ is the same as an A_4-category structure $(m_1 = 0, m_2, m_3)$ in $\mathcal{T}_\Sigma$, with m_2 the composition in $\mathcal{T}_\Sigma$.

An A_∞-category $\mathcal{A}$ consists of

- Objects $X, Y, \dots$
- Morphism $\mathbb{Z}$-graded k-modules $\mathcal{A}(X, Y)$,
- Identity morphisms $\mathrm{id}_X \in \mathcal{A}(X, X)_0$,
- n-Fold composition law, $n \geq 1$,

$$m_n \colon \mathcal{A}(X_1, X_0) \otimes \cdots \otimes \mathcal{A}(X_n, X_{n-1}) \longrightarrow \mathcal{A}(X_n, X_0),$$

$$\deg(m_n) = 2 - n.$$

A_∞-categories

An element $\{m_3\} \in HH^{3,-1}(\mathcal{T}_\Sigma)$ is the same as an A_4-category structure $(m_1 = 0, m_2, m_3)$ in $\mathcal{T}_\Sigma$, with m_2 the composition in $\mathcal{T}_\Sigma$.

An A_∞-category $\mathcal{A}$ consists of

- Objects X, Y, $\dots$
- Morphism $\mathbb{Z}$-graded k-modules $\mathcal{A}(X, Y)$,
- Identity morphisms $\mathrm{id}_X \in \mathcal{A}(X, X)_0$,
- n-Fold composition law, $n \geq 1$,

$$m_n \colon \mathcal{A}(X_1, X_0) \otimes \cdots \otimes \mathcal{A}(X_n, X_{n-1}) \longrightarrow \mathcal{A}(X_n, X_0),$$

$$\deg(m_n) = 2 - n.$$

An element $\{m_3\} \in HH^{3,-1}(\mathcal{T}_\Sigma)$ is the same as an A_4-category structure $(m_1 = 0, m_2, m_3)$ in $\mathcal{T}_\Sigma$, with m_2 the composition in $\mathcal{T}_\Sigma$.

An A_∞-category $\mathcal{A}$ consists of

- Objects X, Y, ...
- Morphism $\mathbb{Z}$-graded k-modules $\mathcal{A}(X, Y)$,
- Identity morphisms $\mathrm{id}_X \in \mathcal{A}(X, X)_0$,
- n-Fold composition law, $n \geq 1$,

$$m_n \colon \mathcal{A}(X_1, X_0) \otimes \cdots \otimes \mathcal{A}(X_n, X_{n-1}) \longrightarrow \mathcal{A}(X_n, X_0),$$

$$\deg(m_n) = 2 - n.$$

A_∞-categories

An element $\{m_3\} \in HH^{3,-1}(\mathcal{T}_\Sigma)$ is the same as an A_4-category structure $(m_1 = 0, m_2, m_3)$ in $\mathcal{T}_\Sigma$, with m_2 the composition in $\mathcal{T}_\Sigma$.

An A_∞-category $\mathcal{A}$ consists of

- Objects $X, Y, \dots$
- Morphism $\mathbb{Z}$-graded k-modules $\mathcal{A}(X, Y)$,
- Identity morphisms $\mathrm{id}_X \in \mathcal{A}(X, X)_0$,
- n-Fold composition law, $n \geq 1$,

$$m_n \colon \mathcal{A}(X_1, X_0) \otimes \cdots \otimes \mathcal{A}(X_n, X_{n-1}) \longrightarrow \mathcal{A}(X_n, X_0),$$

$$\deg(m_n) = 2 - n.$$

An element $\{m_3\} \in HH^{3,-1}(\mathcal{T}_\Sigma)$ is the same as an A_4-category structure $(m_1 = 0, m_2, m_3)$ in $\mathcal{T}_\Sigma$, with m_2 the composition in $\mathcal{T}_\Sigma$.

An A_∞-category $\mathcal{A}$ consists of

- Objects X, Y, ...
- Morphism $\mathbb{Z}$-graded k-modules $\mathcal{A}(X, Y)$,
- Identity morphisms $\mathrm{id}_X \in \mathcal{A}(X, X)_0$,
- n-Fold composition law, $n \geq 1$,

$$m_n \colon \mathcal{A}(X_1, X_0) \otimes \cdots \otimes \mathcal{A}(X_n, X_{n-1}) \longrightarrow \mathcal{A}(X_n, X_0),$$

$$\deg(m_n) = 2 - n.$$

An element $\{m_3\} \in HH^{3,-1}(\mathcal{T}_\Sigma)$ is the same as an A_4-category structure $(m_1 = 0, m_2, m_3)$ in $\mathcal{T}_\Sigma$, with m_2 the composition in $\mathcal{T}_\Sigma$.

An *A_∞-category* $\mathcal{A}$ consists of

- Objects X, Y, ...
- Morphism $\mathbb{Z}$-graded k-modules $\mathcal{A}(X, Y)$,
- Identity morphisms $\mathrm{id}_X \in \mathcal{A}(X, X)_0$,
- n-Fold composition law, $n \geq 1$,

$$m_n \colon \mathcal{A}(X_1, X_0) \otimes \cdots \otimes \mathcal{A}(X_n, X_{n-1}) \longrightarrow \mathcal{A}(X_n, X_0),$$

$$\deg(m_n) = 2 - n.$$

A_∞-categories

The composition laws must satisfy the following equations,

$$0 = \sum_{\substack{j+p+q=n \\ i=j+1+q}} (-1)^{jp+q} m_i(1^{\otimes j} \otimes m_p \otimes 1^{\otimes q}), \qquad n \geq 1.$$

- $n = 1$, $m_1^2 = 0$, i.e. $\mathcal{A}(X, Y)$ are complexes.
- $n = 2$,
$$m_1 m_2 = m_2(1 \otimes m_1 + m_1 \otimes 1),$$
i.e. m_1 is a derivation for the product m_2.
- $n = 3$,
$$m_2(m_2 \otimes 1 - 1 \otimes m_2) = m_1 m_3 + m_3(1 \otimes 1 \otimes m_1 + 1 \otimes m_1 \otimes 1 + m_1 \otimes 1 \otimes 1),$$
i.e. m_2 is associative up to homotopy.

A_∞-categories

The composition laws must satisfy the following equations,

$$0 = \sum_{\substack{j+p+q=n \\ i=j+1+q}} (-1)^{jp+q} m_i(1^{\otimes j} \otimes m_p \otimes 1^{\otimes q}), \qquad n \geq 1.$$

- $n = 1$, $m_1^2 = 0$, i.e. $\mathcal{A}(X, Y)$ are complexes.
- $n = 2$,
$$m_1 m_2 = m_2(1 \otimes m_1 + m_1 \otimes 1),$$
i.e. m_1 is a derivation for the product m_2.
- $n = 3$,
$$m_2(m_2 \otimes 1 - 1 \otimes m_2) = m_1 m_3 + m_3(1 \otimes 1 \otimes m_1 + 1 \otimes m_1 \otimes 1 + m_1 \otimes 1 \otimes 1),$$
i.e. m_2 is associative up to homotopy.

A_∞-categories

The composition laws must satisfy the following equations,

$$0 = \sum_{\substack{j+p+q=n \\ i=j+1+q}} (-1)^{jp+q} m_i(1^{\otimes j} \otimes m_p \otimes 1^{\otimes q}), \qquad n \geq 1.$$

- $n = 1$, $m_1^2 = 0$, i.e. $\mathcal{A}(X, Y)$ are complexes.
- $n = 2$,

  $$m_1 m_2 = m_2(1 \otimes m_1 + m_1 \otimes 1),$$

  i.e. m_1 is a derivation for the product m_2.
- $n = 3$,

  $$m_2(m_2 \otimes 1 - 1 \otimes m_2) = m_1 m_3 + m_3(1 \otimes 1 \otimes m_1 + 1 \otimes m_1 \otimes 1 + m_1 \otimes 1 \otimes 1),$$

  i.e. m_2 is associative up to homotopy.

A_∞-categories

The composition laws must satisfy the following equations,

$$0 = \sum_{\substack{j+p+q=n \\ i=j+1+q}} (-1)^{jp+q} m_i(1^{\otimes j} \otimes m_p \otimes 1^{\otimes q}), \qquad n \geq 1.$$

- $n = 1$, $m_1^2 = 0$, i.e. $\mathcal{A}(X, Y)$ are complexes.
- $n = 2$,
$$m_1 m_2 = m_2(1 \otimes m_1 + m_1 \otimes 1),$$
i.e. m_1 is a derivation for the product m_2.
- $n = 3$,
$$m_2(m_2 \otimes 1 - 1 \otimes m_2) = m_1 m_3 + m_3(1 \otimes 1 \otimes m_1 + 1 \otimes m_1 \otimes 1 + m_1 \otimes 1 \otimes 1),$$
i.e. m_2 is associative up to homotopy.

An A_∞-category is pretriangulated if the full subcategory of the derived category $H^0\mathcal{A} \subset D(\mathcal{A})$ is a triangulated subcategory.

An A_∞-category is minimal if $m_1 = 0$.

Proposition (Lefèvre-Hasegawa'03)

A compactly generated algebraic triangulated k-category $\mathcal{T}$ is $H^0\mathcal{A}$ of a minimal pretringulated A_∞-category $\mathcal{A}$.

The underlying $\mathbb{Z}$-graded k-category of $\mathcal{A}$ is actually $\mathcal{T}_\Sigma$, so in order to reconstruct $\mathcal{A}$ one just has to find $m_3, m_4, \ldots$

An A_∞-category is pretriangulated if the full subcategory of the derived category $H^0\mathcal{A} \subset D(\mathcal{A})$ is a triangulated subcategory.

An A_∞-category is minimal if $m_1 = 0$.

Proposition (Lefèvre-Hasegawa'03)

A compactly generated algebraic triangulated k-category $\mathcal{T}$ is $H^0\mathcal{A}$ of a minimal pretringulated A_∞-category $\mathcal{A}$.

The underlying $\mathbb{Z}$-graded k-category of $\mathcal{A}$ is actually $\mathcal{T}_\Sigma$, so in order to reconstruct $\mathcal{A}$ one just has to find $m_3, m_4, \dots$

An A_∞-category is pretriangulated if the full subcategory of the derived category $H^0\mathcal{A} \subset D(\mathcal{A})$ is a triangulated subcategory.

An A_∞-category is minimal if $m_1 = 0$.

Proposition (Lefèvre-Hasegawa'03)

A compactly generated algebraic triangulated k-category $\mathcal{T}$ is $H^0\mathcal{A}$ of a minimal pretringulated A_∞-category $\mathcal{A}$.

The underlying $\mathbb{Z}$-graded k-category of $\mathcal{A}$ is actually $\mathcal{T}_\Sigma$, so in order to reconstruct $\mathcal{A}$ one just has to find $m_3, m_4, \dots$

A_∞-categories

An A_∞-category is pretriangulated if the full subcategory of the derived category $H^0\mathcal{A} \subset D(\mathcal{A})$ is a triangulated subcategory.

An A_∞-category is minimal if $m_1 = 0$.

Proposition (Lefèvre-Hasegawa'03)

A compactly generated algebraic triangulated k-category $\mathcal{T}$ is $H^0\mathcal{A}$ of a minimal pretringulated A_∞-category $\mathcal{A}$.

The underlying $\mathbb{Z}$-graded k-category of $\mathcal{A}$ is actually $\mathcal{T}_\Sigma$, so in order to reconstruct $\mathcal{A}$ one just has to find $m_3, m_4, \dots$

The existence of m_3 is equivalent to say that the Massey product of $\mathcal{T}$ is in the image of the composite

$$HH^{3,-1}(\mathcal{T}_\Sigma) \longrightarrow HH^{3,-1}(\mathcal{T}) \xrightarrow{\text{edge}} HH^0(\text{mod-}\mathcal{T}, \operatorname{Ext}^{3,-1}_{\mathcal{T}}) \cong \mathrm{MP}(\mathcal{T}, \Sigma).$$

In order to check this fact, one can use the spectral sequence

$$HH^p(\text{mod-}\mathcal{T}, \operatorname{Ext}^{q,r}_{\mathcal{T}}) \Longrightarrow HH^{p+q,r}(\mathcal{T})$$

and the long exact sequence

$$\cdots \to HH^{p,q}(\mathcal{T}_\Sigma) \to HH^{p,q}(\mathcal{T}) \xrightarrow{\Sigma_*^{-1}\Sigma^*+1} HH^{p,q}(\mathcal{T}) \to HH^{p+1,q}(\mathcal{T}_\Sigma) \to \cdots$$

The existence of m_3 is equivalent to say that the Massey product of $\mathcal{T}$ is in the image of the composite

$$HH^{3,-1}(\mathcal{T}_\Sigma) \longrightarrow HH^{3,-1}(\mathcal{T}) \xrightarrow{\text{edge}} HH^0(\text{mod-}\mathcal{T}, \text{Ext}^{3,-1}_{\mathcal{T}}) \cong \text{MP}(\mathcal{T}, \Sigma).$$

In order to check this fact, one can use the spectral sequence

$$HH^p(\text{mod-}\mathcal{T}, \text{Ext}^{q,r}_{\mathcal{T}}) \Longrightarrow HH^{p+q,r}(\mathcal{T})$$

and the long exact sequence

$$\cdots \to HH^{p,q}(\mathcal{T}_\Sigma) \to HH^{p,q}(\mathcal{T}) \xrightarrow{\Sigma_*^{-1}\Sigma^*+1} HH^{p,q}(\mathcal{T}) \to HH^{p+1,q}(\mathcal{T}_\Sigma) \to \cdots$$

Lemma (Lefèvre-Hasegawa'03)

Let $n \geq 5$. Given a minimal A_{n-1}-category structure on $\mathcal{T}_\Sigma$, defined by

$$(m_1 = 0, m_2, m_3, \dots, m_{n-2}),$$

there is a well-defined

$$\theta_{(m_3,\dots,m_{n-2})} \in HH^{n,3-n}(\mathcal{T}_\Sigma),$$

which vanishes if and only if there exists m_{n-1} such that

$$(m_1 = 0, m_2, m_3, \dots, m_{n-2}, m_{n-1})$$

is an A_n-category structure on $\mathcal{T}_\Sigma$.

Summing up the A_∞-obstruction theory

Let $\mathcal{T}$ be triangulated $\leftrightsquigarrow$ $\delta \in H^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}})$.

- δ must be a perm. cycle of $HH^p(\text{mod-}\mathcal{T}, \mathrm{Ext}^{q,-1}_{\mathcal{T}}) \Rightarrow HH^{p+q,-1}(\mathcal{T})$,

$$\begin{array}{ccc} HH^{3,-1}(\mathcal{T}) & \xrightarrow{\text{edge}} & H^0(\text{mod-}\mathcal{T}, \mathrm{Ext}^{3,-1}_{\mathcal{T}}), \\ \Delta & \mapsto & \delta. \end{array}$$

- Δ must be in the kernel of $HH^{3,-1}(\mathcal{T}) \xrightarrow{\Sigma_*^{-1}\Sigma^*+1} HH^{3,-1}(\mathcal{T})$, so

$$\begin{array}{ccc} HH^{3,-1}(\mathcal{T}_\Sigma) & \longrightarrow & HH^{3,-1}(\mathcal{T}), \\ \{m_3\} & \mapsto & \Delta. \end{array}$$

- The higher obstructions must vanish,

$$\theta_{(m_3,\dots,m_{n-2})} \in H^{n,3-n}(\mathcal{T}_\Sigma), \quad n \geq 5.$$

Then $\mathcal{T}$ can be enhanced to an A_∞-category defined over $\mathcal{T}_\Sigma$.

Summing up the A_∞-obstruction theory

Let $\mathcal{T}$ be triangulated $\leftrightsquigarrow$ $\delta \in H^0(\text{mod-}\mathcal{T}, \operatorname{Ext}^{3,-1}_{\mathcal{T}})$.

- δ must be a perm. cycle of $HH^p(\text{mod-}\mathcal{T}, \operatorname{Ext}^{q,-1}_{\mathcal{T}}) \Rightarrow HH^{p+q,-1}(\mathcal{T})$,

$$\begin{array}{ccc} HH^{3,-1}(\mathcal{T}) & \xrightarrow{\text{edge}} & H^0(\text{mod-}\mathcal{T}, \operatorname{Ext}^{3,-1}_{\mathcal{T}}), \\ \Delta & \mapsto & \delta. \end{array}$$

- Δ must be in the kernel of $HH^{3,-1}(\mathcal{T}) \xrightarrow{\Sigma_*^{-1}\Sigma^*+1} HH^{3,-1}(\mathcal{T})$, so

$$\begin{array}{ccc} HH^{3,-1}(\mathcal{T}_\Sigma) & \longrightarrow & HH^{3,-1}(\mathcal{T}), \\ \{m_3\} & \mapsto & \Delta. \end{array}$$

- The higher obstructions must vanish,

$$\theta_{(m_3,\dots,m_{n-2})} \in H^{n,3-n}(\mathcal{T}_\Sigma), \quad n \geq 5.$$

Then $\mathcal{T}$ can be enhanced to an A_∞-category defined over $\mathcal{T}_\Sigma$.

Summing up the A_∞-obstruction theory

Let $\mathcal{T}$ be triangulated $\leftrightsquigarrow$ $\delta \in H^0(\text{mod-}\mathcal{T}, \text{Ext}_{\mathcal{T}}^{3,-1})$.

- δ must be a perm. cycle of $HH^p(\text{mod-}\mathcal{T}, \text{Ext}_{\mathcal{T}}^{q,-1}) \Rightarrow HH^{p+q,-1}(\mathcal{T})$,

$$\begin{array}{ccc} HH^{3,-1}(\mathcal{T}) & \xrightarrow{\text{edge}} & H^0(\text{mod-}\mathcal{T}, \text{Ext}_{\mathcal{T}}^{3,-1}), \\ \Delta & \mapsto & \delta. \end{array}$$

- Δ must be in the kernel of $HH^{3,-1}(\mathcal{T}) \xrightarrow{\Sigma_*^{-1}\Sigma^*+1} HH^{3,-1}(\mathcal{T})$, so

$$\begin{array}{ccc} HH^{3,-1}(\mathcal{T}_\Sigma) & \longrightarrow & HH^{3,-1}(\mathcal{T}), \\ \{m_3\} & \mapsto & \Delta. \end{array}$$

- The higher obstructions must vanish,

$$\theta_{(m_3,\dots,m_{n-2})} \in H^{n,3-n}(\mathcal{T}_\Sigma), \quad n \geq 5.$$

Then $\mathcal{T}$ can be enhanced to an A_∞-category defined over $\mathcal{T}_\Sigma$.

Summing up the A_∞-obstruction theory

Let $\mathcal{T}$ be triangulated $\leftrightsquigarrow \delta \in H^0(\text{mod-}\mathcal{T}, \operatorname{Ext}^{3,-1}_{\mathcal{T}})$.

- δ must be a perm. cycle of $HH^p(\text{mod-}\mathcal{T}, \operatorname{Ext}^{q,-1}_{\mathcal{T}}) \Rightarrow HH^{p+q,-1}(\mathcal{T})$,

$$\begin{array}{ccc} HH^{3,-1}(\mathcal{T}) & \xrightarrow{\text{edge}} & H^0(\text{mod-}\mathcal{T}, \operatorname{Ext}^{3,-1}_{\mathcal{T}}), \\ \Delta & \mapsto & \delta. \end{array}$$

- Δ must be in the kernel of $HH^{3,-1}(\mathcal{T}) \xrightarrow{\Sigma_*^{-1}\Sigma^*+1} HH^{3,-1}(\mathcal{T})$, so

$$\begin{array}{ccc} HH^{3,-1}(\mathcal{T}_\Sigma) & \longrightarrow & HH^{3,-1}(\mathcal{T}), \\ \{m_3\} & \mapsto & \Delta. \end{array}$$

- The higher obstructions must vanish,

$$\theta_{(m_3,\dots,m_{n-2})} \in H^{n,3-n}(\mathcal{T}_\Sigma), \quad n \geq 5.$$

Then $\mathcal{T}$ can be enhanced to an A_∞-category defined over $\mathcal{T}_\Sigma$.

Let $\mathcal{T}$ be triangulated $\leftrightsquigarrow$ $\delta \in H^0(\text{mod-}\mathcal{T}, \operatorname{Ext}^{3,-1}_{\mathcal{T}})$.

- δ must be a perm. cycle of $HH^p(\text{mod-}\mathcal{T}, \operatorname{Ext}^{q,-1}_{\mathcal{T}}) \Rightarrow HH^{p+q,-1}(\mathcal{T})$,

$$\begin{array}{ccc} HH^{3,-1}(\mathcal{T}) & \xrightarrow{\text{edge}} & H^0(\text{mod-}\mathcal{T}, \operatorname{Ext}^{3,-1}_{\mathcal{T}}), \\ \Delta & \mapsto & \delta. \end{array}$$

- Δ must be in the kernel of $HH^{3,-1}(\mathcal{T}) \xrightarrow{\Sigma_*^{-1}\Sigma^*+1} HH^{3,-1}(\mathcal{T})$, so

$$\begin{array}{ccc} HH^{3,-1}(\mathcal{T}_\Sigma) & \longrightarrow & HH^{3,-1}(\mathcal{T}), \\ \{m_3\} & \mapsto & \Delta. \end{array}$$

- The higher obstructions must vanish,

$$\theta_{(m_3,\dots,m_{n-2})} \in H^{n,3-n}(\mathcal{T}_\Sigma), \quad n \geq 5.$$

Then $\mathcal{T}$ can be enhanced to an A_∞-category defined over $\mathcal{T}_\Sigma$.

Let $\mathcal{T}$ be triangulated $\leftrightsquigarrow$ $\delta \in H^0(\text{mod-}\mathcal{T}, \mathrm{Ext}_{\mathcal{T}}^{3,-1})$.

- δ must be a perm. cycle of $HH^p(\text{mod-}\mathcal{T}, \mathrm{Ext}_{\mathcal{T}}^{q,-1}) \Rightarrow HH^{p+q,-1}(\mathcal{T})$,

$$\begin{array}{ccc} HH^{3,-1}(\mathcal{T}) & \xrightarrow{\text{edge}} & H^0(\text{mod-}\mathcal{T}, \mathrm{Ext}_{\mathcal{T}}^{3,-1}), \\ \Delta & \mapsto & \delta. \end{array}$$

- Δ must be in the kernel of $HH^{3,-1}(\mathcal{T}) \xrightarrow{\Sigma_*^{-1}\Sigma^*+1} HH^{3,-1}(\mathcal{T})$, so

$$\begin{array}{ccc} HH^{3,-1}(\mathcal{T}_\Sigma) & \longrightarrow & HH^{3,-1}(\mathcal{T}), \\ \{m_3\} & \mapsto & \Delta. \end{array}$$

- The higher obstructions must vanish,

$$\theta_{(m_3,\dots,m_{n-2})} \in H^{n,3-n}(\mathcal{T}_\Sigma), \quad n \geq 5.$$

Then $\mathcal{T}$ can be enhanced to an A_∞-category defined over $\mathcal{T}_\Sigma$.

Open questions

- Find a $\mathbb{Q}$-linear triangulated category $\mathcal{T}$ with non-vanishing A_∞-obstructions.
- Extend the A_∞-obstruction theory to an arbitrary commutative ground ring k (by using Shukla cohomology).
- Extend the A_∞-obstruction theory to spectral categories (by using topological Hochschild cohomology).

Open questions

- Find a $\mathbb{Q}$-linear triangulated category $\mathcal{T}$ with non-vanishing A_∞-obstructions.
- Extend the A_∞-obstruction theory to an arbitrary commutative ground ring k (by using Shukla cohomology).
- Extend the A_∞-obstruction theory to spectral categories (by using topological Hochschild cohomology).

Open questions

- Find a $\mathbb{Q}$-linear triangulated category $\mathcal{T}$ with non-vanishing A_∞-obstructions.
- Extend the A_∞-obstruction theory to an arbitrary commutative ground ring k (by using Shukla cohomology).
- Extend the A_∞-obstruction theory to spectral categories (by using topological Hochschild cohomology).