## On Massey products and triangulated categories

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Homotopy Theory and Higher Categories 2007–2008 Workshop on Derived Categories

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## • Massey products and Heller's theory.

- Cohomology of categories and Massey products.
- Stable Massey products and  $A_{\infty}$ -enhancements.

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#### Let k be a commutative ring.

Let T be a k-linear, additive, idempotent complete category.

A (right)  $\mathcal{T}$ -module *M* is a *k*-linear functor  $M: \mathcal{T}^{op} \to Mod$ -*k*. It is finitely presented or coherent if there exists an exact sequence

$$\mathcal{T}(-,X)\longrightarrow \mathcal{T}(-,Y)\longrightarrow M\rightarrow 0.$$

Let mod-T be the category of coherent T-modules.

## Theorem (Freyd'66)

If T is triangulated then mod- T is a Frobenius abelian category and T is the full subcategory of injective-projective objects.

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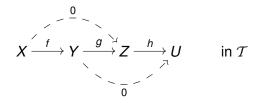
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A Massey product or secondary composition sends



to

$$\langle h, g, f \rangle \subset \mathcal{T}(\Sigma X, U),$$

a coset of

 $h \cdot \mathcal{T}(\Sigma X, Z) + \mathcal{T}(\Sigma Y, U) \cdot (\Sigma f) \subset \mathcal{T}(\Sigma X, U),$ 

the indeterminacy submodule.

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Moreover, given composable morphisms (without vanishing conditions)

$$X \stackrel{f}{\longrightarrow} Y \stackrel{g}{\longrightarrow} Z \stackrel{h}{\longrightarrow} U \stackrel{i}{\longrightarrow} V,$$

the following inclusions hold whenever the Massey products are defined,

$$\langle i,h,g \rangle \cdot (\Sigma f) \subset \langle i,h,g \cdot f \rangle \subset \langle i,h \cdot g,f \rangle \supset \langle i \cdot h,g,f \rangle \supset i \cdot \langle h,g,f \rangle.$$

The set of Massey products is a *k*-module,

 $\mathsf{MP}(\mathcal{T}, \Sigma).$ 

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### Theorem (Heller'68)

If  $\mathcal{T}$  is triangulated there is a unique Massey product such that for any exact triangle

$$X \stackrel{f}{\longrightarrow} Y \stackrel{i}{\longrightarrow} C \stackrel{q}{\longrightarrow} \Sigma X$$

we have

$$1_{\Sigma X} \in \langle q, i, f \rangle \subset \mathcal{T}(\Sigma X, \Sigma X).$$

This defines an inclusion

 $\{triangulated structures on (T, \Sigma)\} \subset MP(T, \Sigma).$ 

skip proof

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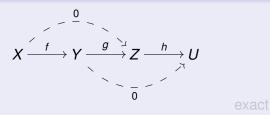
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Idea of the proof.



A triangle  $X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{q} \Sigma X$  is exact if and only if

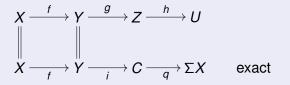
$$\mathcal{T}(-,X) \xrightarrow{f_*} \mathcal{T}(-,Y) \xrightarrow{i_*} \mathcal{T}(-,C) \xrightarrow{q_*} \mathcal{T}(-,\Sigma X) \xrightarrow{(\Sigma f)_*} \mathcal{T}(-,\Sigma Y)$$

is an exact sequence of  $\mathcal{T}$ -modules and  $1_{\Sigma X} \in \langle q, i, f \rangle$ .

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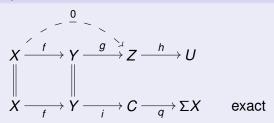
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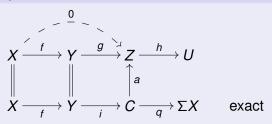
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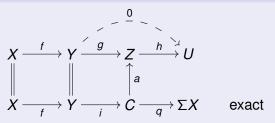
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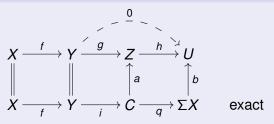
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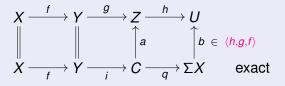
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$$\begin{array}{cccc} X & \stackrel{f}{\longrightarrow} Y & \stackrel{g}{\longrightarrow} Z & \stackrel{h}{\longrightarrow} U \\ & & & & \uparrow a & \uparrow b \in \langle h, g, f \rangle \\ X & \stackrel{f}{\longrightarrow} Y & \stackrel{f}{\longrightarrow} C & \stackrel{g}{\longrightarrow} \Sigma X & \text{exact} \end{array}$$

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#### When is a Massey product induced by a triangulated structure?

Let <u>mod</u>- $\mathcal T$  be the stable category of coherent  $\mathcal T$ -modules,

$$\underline{\operatorname{Hom}}_{\mathcal{T}}(M,N) = \frac{\operatorname{Hom}_{\mathcal{T}}(M,N)}{\{M \to \mathcal{T}(-,X) \to N\}}.$$

The stable category is triangulated. The translation functor

$$S: \underline{\mathsf{mod}} \ \mathcal{T} \longrightarrow \underline{\mathsf{mod}} \ \mathcal{T}$$

is determined by the choice of short exact sequences in mod- $\mathcal{T}$ ,

$$0 \to M \longrightarrow \mathcal{T}(-, CM) \longrightarrow SM \to 0.$$

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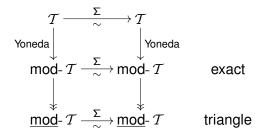
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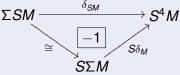
The functor  $\Sigma$  extends in an essentially unique way,



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#### Theorem (Heller'68)

There is a bijective correspondence between Puppe triangulated structures on  $(\mathcal{T}, \Sigma)$  and natural isomorphisms  $\delta \colon \Sigma \cong S^3$  such that for any coherent  $\mathcal{T}$ -module M,



#### Theorem

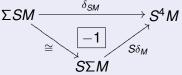
There is an isomorphism which sends the Massey product of a triangulation on  $(\mathcal{T}, \Sigma)$  to Heller's natural isomorphism,

## $\mathsf{MP}(\mathcal{T}, \Sigma) \cong \mathsf{Hom}(\Sigma, S^3)$

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## Heller's theory

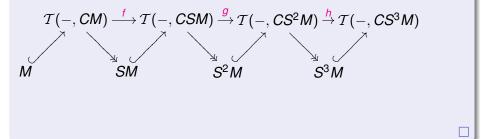
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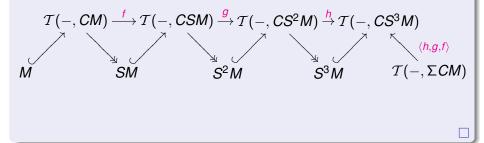
Let  $\langle -, -, - \rangle$  be a Massey product. We need to define a morphism  $\delta_M \colon \Sigma M \to S^3 M$  for any coherent  $\mathcal{T}$ -module M.

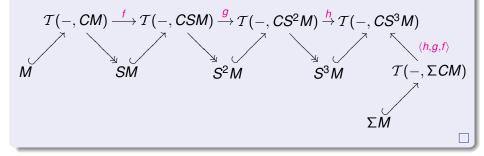


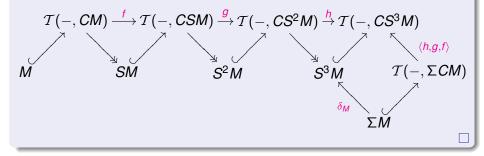
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## An example of Heller's theory

Let  $\mathcal{T} = \mathcal{F}(\mathbb{Z}/4)$  be the category of finitely generated free  $\mathbb{Z}/4$ -modules and  $\Sigma = 1_{\mathcal{F}(\mathbb{Z}/4)}$  the identity functor.

In this case mod-  $\mathcal{T} =$  mod-  $\mathbb{Z}/4$ , mod-  $\mathcal{T} = \mathcal{F}(\mathbb{Z}/2)$  and  $S = 1_{\mathcal{F}(\mathbb{Z}/2)}$ .

 $\mathsf{MP}(\mathcal{F}(\mathbb{Z}/4), \mathbf{1}_{\mathcal{F}(\mathbb{Z}/4)}) \cong \mathsf{Hom}(\mathbf{1}_{\mathcal{F}(\mathbb{Z}/2)}, \mathbf{1}_{\mathcal{F}(\mathbb{Z}/2)}) \cong \mathbb{Z}/2.$ 

Theorem (M.-Schwede-Strickland'07)

The non-trivial Massey product in  $(\mathcal{F}(\mathbb{Z}/4), 1_{\mathcal{F}(\mathbb{Z}/4)})$  is induced by a Verdier triangulated structure where the triangle

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#### A $\mathcal{T}$ -bimodule is a $\mathcal{T} \otimes \mathcal{T}^{op}$ -module.

The bar complex  $C_*(\mathcal{T})$  is the complex of  $\mathcal{T}$ -bimodules

$$C_*(\mathcal{T}) = \bigoplus_{X_0,...,X_n} \mathcal{T}(X_0,-) \otimes \cdots \otimes \mathcal{T}(X_i,X_{i-1}) \otimes \cdots \otimes \mathcal{T}(-,X_n),$$

with differential

$$\partial(\alpha_0\otimes\cdots\otimes\alpha_{n+1}) = \sum_{i=0}^n (-1)^i \alpha_0\otimes\cdots\otimes(\alpha_i\alpha_{i+1})\otimes\cdots\otimes\alpha_{n+1}.$$

The Hochschild-Mitchell cohomology of  $\mathcal T$  with coefficients in M,

 $HH^*(\mathcal{T}, M),$ 

is the cohomology of

$$C^*(\mathcal{T}, M) = \operatorname{Hom}_{\mathcal{T}-\operatorname{bimod}}(C_*(\mathcal{T}), M).$$

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#### Example

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 $\operatorname{Ext}_{\mathcal{T}}^{q,r} = \operatorname{Ext}_{\mathcal{T}}^{q}(-,\Sigma^{r}) \cong \operatorname{Ext}_{\mathcal{T}}^{q}(\Sigma^{-r},-), \quad q \geq 0, \ r \in \mathbb{Z},$ 

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#### **Baues-Wirsching cohomology**

#### The Baues-Wirsching cohomology of T with coefficients in M,

 $H^*(\mathcal{T}, M),$ 

is the cohomology of the 'group ring' k-category k[T] obtained by taking free k-modules on morphism pointed sets,

 $k[\mathcal{T}](X, Y) =$  free k-module on  $\mathcal{T}(X, Y)$ .

The natural k-linear functor  $k[\mathcal{T}] o \mathcal{T}$  induces a homomorphism

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A Baues-Wirsching (3, -1)-cocycle  $z_{3,-1}$  of T sends any three composable morphisms

$$X \stackrel{f}{\longrightarrow} Y \stackrel{g}{\longrightarrow} Z \stackrel{h}{\longrightarrow} U$$

to an element

$$z_{3,-1}(h,g,f) \in \mathcal{T}(\Sigma X,U),$$

in such a way that

$$i \cdot z_{3,-1}(h,g,f) - z_{3,-1}(i \cdot h,g,f) + z_{3,-1}(i,h \cdot g,f) - z_{3,-1}(i,h,g \cdot f) + z_{3,-1}(i,h,g) \cdot (\Sigma f) = 0.$$

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Given a Baues-Wirsching (3, -1)-cocycle  $z_{3,-1}$  there is defined a unique Massey product in  $(\mathcal{T}, \Sigma)$  such that

 $z_{3,-1}(h,g,f) \in \langle h,g,f \rangle \subset \mathcal{T}(\Sigma X,U).$ 

This defines a homomorphism

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#### Theorem (Pirashvili'88, Baues-Dreckmann'89)

The Massey product of a topological triangulated category is in the image of

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The Massey product of a locally projective algebraic triangulated category is in the image of

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skip proof

#### Is there any triangulated category whose Massey product does not come from *HH*<sup>3,-1</sup> or *H*<sup>3,-1</sup>?

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#### Idea of the proof.

Let  $\mathcal{M}$  be a topological or algebraic model of  $\mathcal{T}$  such that  $\mathcal{T} \subset D(\mathcal{M})$  as a full triangulated subcategory. There is defined a derived 2-category  $D_2(\mathcal{M})$ , and a projection

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The obstruction to the existence of a splitting pseudofunctor is

 $\langle D_2(\mathcal{M}) \rangle_{|_{\mathcal{T}}} \in H^{3,-1}(\mathcal{T})$  universal Massey product

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There is an isomorphism

$$\mathsf{MP}(\mathcal{T}, \Sigma) \cong H^0(\mathsf{mod}_{\mathcal{T}}, \mathsf{Ext}^{3, -1}_{\mathcal{T}}).$$

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$$\begin{aligned} \mathsf{MP}(\mathcal{T}, \Sigma) &\cong & \mathsf{Hom}(\Sigma, S^3) \\ &\cong & H^0(\operatorname{mod}_{-} \mathcal{T}, \operatorname{\underline{Hom}}_{\mathcal{T}}(\Sigma, S^3)) \\ &\cong & H^0(\operatorname{mod}_{-} \mathcal{T}, \operatorname{Ext}^{3, -1}_{\mathcal{T}}), \end{aligned}$$

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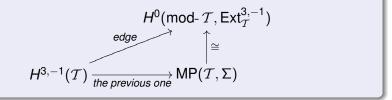
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#### For $(\mathcal{F}(\mathbb{Z}/4), 1_{\mathcal{F}(\mathbb{Z}/4)})$ the edge homomorphism is trivial.

 $\mathbb{Z}/2 \cong HML^{3}(\mathbb{Z}/4) \cong H^{3,-1}(\mathcal{F}(\mathbb{Z}/4)) \stackrel{0}{\longrightarrow} H^{0}(\text{mod-} \mathbb{Z}/4, \text{Ext}^{3,-1}_{\mathbb{Z}/4}) \cong \mathbb{Z}/2.$ 

#### Corollary (M.-Schwede-Strickland'07)

The triangulated category  $\mathcal{F}(\mathbb{Z}/4)$  does not have any algebraic or topological model.

When does a triangulated category have a model? Is there an obstruction theory for the existence of models of any kind?

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#### A Massey product on $(\mathcal{T}, \Sigma)$ is stable if

 $\langle \Sigma h, \Sigma g, \Sigma f \rangle = -\Sigma \langle h, g, f \rangle.$ 

Therefore the submodule of stable Massey products  $MP_{\mathcal{S}}(\mathcal{T},\Sigma)$  is the kernel of

$$\mathsf{MP}(\mathcal{T}, \Sigma) \cong HH^0(\mathsf{mod-}\mathcal{T}, \mathsf{Ext}_{\mathcal{T}}^{3,-1}) \xrightarrow{\Sigma_*^{-1}\Sigma^*+1} HH^0(\mathsf{mod-}\mathcal{T}, \mathsf{Ext}_{\mathcal{T}}^{3,-1}).$$

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#### Let *k* be a field and $\mathcal{T}_{\Sigma}$ the $\mathbb{Z}$ -graded *k*-category with

 $\mathcal{T}_{\Sigma}(X,Y)^n = \mathcal{T}(X,\Sigma^nY), \quad n \in \mathbb{Z}.$ 

A  $\mathcal{T}_{\Sigma}$ -bimodule is a degree 0 functor  $\mathcal{T}_{\Sigma}^{op} \otimes \mathcal{T}_{\Sigma} \to Mod^{\mathbb{Z}}$ - k to  $\mathbb{Z}$ -graded k-modules.

The bar complex  $\mathcal{C}_*(\mathcal{T}_{\Sigma})$  is now a complex of  $\mathcal{T}_{\Sigma}$ -bimodules.

Given a  $T_{\Sigma}$ -bimodule *M* the Hochschild-Mitchell cohomology

 $HH^{p,q}(\mathcal{T}_{\Sigma}, M),$ 

is the p<sup>th</sup> cohomology of

 $C^*(\mathcal{T}_{\Sigma}, M[q]) = \operatorname{Hom}_{\mathcal{T}_{\Sigma} \operatorname{-bimod}}(C_*(\mathcal{T}_{\Sigma}), M[q]).$ 

#### Example

 $\mathcal{T}_{\Sigma} = \mathcal{T}_{\Sigma}(-,-)$  is a  $\mathcal{T}_{\Sigma}$ -bimodule and  $HH^{p,q}(\mathcal{T}_{\Sigma}) = HH^{p,q}(\mathcal{T}_{\Sigma},\mathcal{T}_{\Sigma})$ .

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A  $\mathcal{T}_{\Sigma}$ -bimodule is a degree 0 functor  $\mathcal{T}_{\Sigma}^{op} \otimes \mathcal{T}_{\Sigma} \to Mod^{\mathbb{Z}}$ - k to  $\mathbb{Z}$ -graded k-modules.

The bar complex  $\mathcal{C}_*(\mathcal{T}_\Sigma)$  is now a complex of  $\mathcal{T}_\Sigma$ -bimodules.

Given a  $\mathcal{T}_{\Sigma}$ -bimodule *M* the Hochschild-Mitchell cohomology

 $HH^{p,q}(\mathcal{T}_{\Sigma}, M),$ 

is the p<sup>th</sup> cohomology of

 $C^*(\mathcal{T}_{\Sigma}, M[q]) = \operatorname{Hom}_{\mathcal{T}_{\Sigma} \operatorname{-bimod}}(C_*(\mathcal{T}_{\Sigma}), M[q]).$ 

#### Example

 $\mathcal{T}_{\Sigma} = \mathcal{T}_{\Sigma}(-,-)$  is a  $\mathcal{T}_{\Sigma}$ -bimodule and  $HH^{p,q}(\mathcal{T}_{\Sigma}) = HH^{p,q}(\mathcal{T}_{\Sigma},\mathcal{T}_{\Sigma})$ .

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#### Proposition

For any  $q \in \mathbb{Z}$ , the complex  $C^*(\mathcal{T}_{\Sigma}, \mathcal{T}_{\Sigma}[q])$  is the homotopy fiber of

$$\Sigma^{-1}_*\Sigma^* + 1 \colon C^*(\mathcal{T}, \mathcal{T}(-, \Sigma^q)) \longrightarrow C^*(\mathcal{T}, \mathcal{T}(-, \Sigma^q)).$$

This homotopy fiber is strongly related to the stability equation for Massey products,

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#### Corollary

There is a long exact sequence for any  $q \in \mathbb{Z}$ ,

$$\cdots o HH^{p,q}(\mathcal{T}_{\Sigma}) o HH^{p,q}(\mathcal{T}) \stackrel{\Sigma_*^{-1}\Sigma^*+1}{\longrightarrow} HH^{p,q}(\mathcal{T}) o HH^{p+1,q}(\mathcal{T}_{\Sigma}) o \cdots$$

Moreover, there is a commutative diagram

$$\begin{array}{c} HH^{3,-1}(\mathcal{T}) \xrightarrow{edge} & HH^{0}(\mathsf{mod-}\mathcal{T},\mathsf{Ext}_{\mathcal{T}}^{3,-1}) \cong \mathsf{MP}(\mathcal{T},\Sigma) \\ & \uparrow \\ & & \uparrow \\ & & & \downarrow \\ HH^{3,-1}(\mathcal{T}_{\Sigma}) \xrightarrow{} \mathsf{MP}_{s}(\mathcal{T},\Sigma) \end{array}$$

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An element  $\{m_3\} \in HH^{3,-1}(\mathcal{T}_{\Sigma})$  is the same as an  $A_4$ -category structure  $(m_1 = 0, m_2, m_3)$  in  $\mathcal{T}_{\Sigma}$ , with  $m_2$  the composition in  $\mathcal{T}_{\Sigma}$ .

An  $A_{\infty}$ -category  $\mathcal{A}$  consists of

- Objects *X*, *Y*, ...
- Morphism  $\mathbb{Z}$ -graded *k*-modules  $\mathcal{A}(X, Y)$ ,
- Identity morphisms  $id_X \in \mathcal{A}(X, X)_0$ ,
- *n*-Fold composition law,  $n \ge 1$ ,

 $m_n: \mathcal{A}(X_1, X_0) \otimes \cdots \otimes \mathcal{A}(X_n, X_{n-1}) \longrightarrow \mathcal{A}(X_n, X_0),$ 

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$$0 = \sum_{\substack{j+p+q=n\\i=j+1+q}} (-1)^{jp+q} m_i (1^{\otimes j} \otimes m_p \otimes 1^{\otimes q}), \qquad n \ge 1.$$

$$m_1m_2=m_2(1\otimes m_1+m_1\otimes 1),$$

i.e.  $m_1$  is a derivation for the product  $m_2$ .

● *n* = 3,

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Proposition (Lefèvre-Hasegawa'03)

A compactly generated algebraic triangulated k-category T is H<sup>0</sup>A of a minimal pretringulated  $A_{\infty}$ -category A.

The underlying  $\mathbb{Z}$ -graded *k*-category of  $\mathcal{A}$  is actually  $\mathcal{T}_{\Sigma}$ , so in order to reconstruct  $\mathcal{A}$  one just has to find  $m_3, m_4, \ldots$ 

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#### $A_{\infty}$ -obstructions for triangulated categories

The existence of  $m_3$  is equivalent to say that the Massey product of T is in the image of the composite

$$H\!H^{3,-1}(\mathcal{T}_{\Sigma}) \longrightarrow H\!H^{3,-1}(\mathcal{T}) \stackrel{\text{edge}}{\longrightarrow} H\!H^{0}(\text{mod-}\mathcal{T},\text{Ext}_{\mathcal{T}}^{3,-1}) \cong \mathsf{MP}(\mathcal{T},\Sigma).$$

In order to check this fact, one can use the spectral sequence

$$HH^{p}(\operatorname{mod-}\mathcal{T},\operatorname{Ext}_{\mathcal{T}}^{q,r}) \Longrightarrow HH^{p+q,r}(\mathcal{T})$$

and the long exact sequence

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### $A_{\infty}$ -obstructions for triangulated categories

#### Lemma (Lefèvre-Hasegawa'03)

Let  $n \ge 5$ . Given a minimal  $A_{n-1}$ -category structure on  $\mathcal{T}_{\Sigma}$ , defined by

$$(m_1 = 0, m_2, m_3, \ldots, m_{n-2}),$$

there is a well-defined

$$\theta_{(m_3,\ldots,m_{n-2})} \in HH^{n,3-n}(\mathcal{T}_{\Sigma}),$$

which vanishes if and only if there exists  $m_{n-1}$  such that

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Let  $\mathcal{T}$  be triangulated  $\iff \delta \in H^0(\text{mod-}\mathcal{T}, \text{Ext}^{3,-1}_{\mathcal{T}}).$ 

•  $\delta$  must be a perm. cycle of  $HH^{p}(\text{mod-}\mathcal{T}, \text{Ext}_{\mathcal{T}}^{q,-1}) \Rightarrow HH^{p+q,-1}(\mathcal{T}),$ 

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• The higher obstructions must vanish,

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