Categorical groups in brave new algebra

Fernando Muro

Universitat de Barcelona
Dept. Àlgebra i Geometria

(joint work with H.-J. Baues)

Workshop on Categorical Groups
Barcelona, June 2008

The stable homotopy category

- It is a symmetric monoidal triangulated category **S** whose objects are called spectra.
- It maps onto the category of cohomology theories for finite CW-complexes.
- Monoids in **S** yield multiplicative cohomology theories.
- $\mathbf{S} = \operatorname{Ho}\mathbf{M}$ for many stable model categories **M**.
- There are symmetric monoidal models **M**. This reflects the existence of higher operations on multiplicative cohomology theories.
- A ring spectrum is a monoid in **M**.
- The homotopy groups of a spectrum E are the cohomology of the point $\pi_*(E) = E^*(\text{pt.})$, and E is connective if $\pi_n(E) = 0$ for $n < 0$.

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Symmetric monoidal categories and stable homotopy

$$\begin{tikzcd}\boxed{\text{Sym. mon. cat.}} \arrow[r, "B", "\text{May, Segal, Thomason}"'] & \boxed{\text{Spectra}}\end{tikzcd}$$

Symmetric monoidal categories and stable homotopy

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Example

- *B(finite sets, bijections, $\coprod$) $=$ S the sphere spectrum.*
- *For R a ring, B(f. g. free left R-mod., iso., $\oplus$) $= K(R)$ the K-theory spectrum of R.*

Symmetric monoidal categories and stable homotopy

$$\begin{tikzcd}
\boxed{\text{Picard groupoids}} \arrow[d] & \\
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\end{tikzcd}$$

Symmetric monoidal categories and stable homotopy

$$\begin{tikzcd}
\text{honest equiv.} \arrow[d, bend left] & & \\
\boxed{\text{Picard groupoids}} \arrow[d] & & \\
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Symmetric monoidal categories and stable homotopy

$$\begin{tikzcd}
\boxed{\text{Picard groupoids}} \arrow[d] \arrow[rr, bend left, "\begin{array}{c}\text{spectra with}\\ \pi_n = 0 \text{ for } n \neq 0, 1\end{array}"] & & \\
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Symmetric monoidal categories and stable homotopy

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\text{Sym. mon. cat.} \arrow[r, "B", "\text{May, Segal, Thomason}"'] & \text{Spectra} \arrow[ul, bend right=60, "\text{Fund. Picard grd.}"']
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$$\begin{tikzcd}
\boxed{\text{Ring categories}} \arrow[r, "B", "\text{Elmendorf–Mandell'06}"'] & \boxed{\text{Ring spectra}}
\end{tikzcd}$$

Symmetric monoidal categories and stable homotopy

$$\begin{tikzcd}
\text{Picard groupoids} \arrow[d] \arrow[rr, bend left=15, "\substack{\text{spectra with}\\ \pi_n = 0 \text{ for } n \neq 0,1}"'] & & \text{Spectra} \arrow[ll, bend right=40, "\text{Fund. Picard grd.}"'] \\
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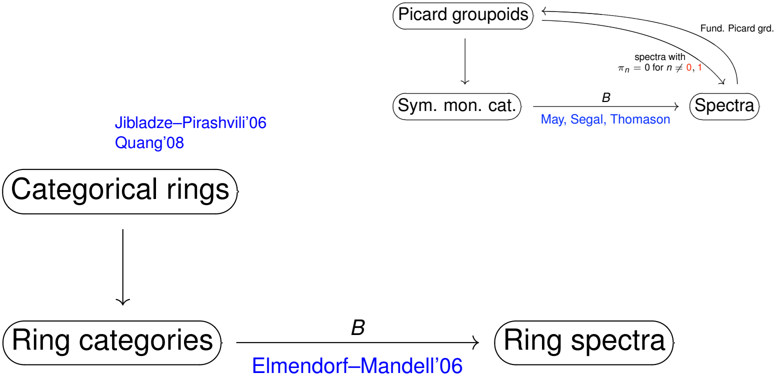
Example

- *B(finite sets, bijections, $\coprod, \times$) $=$ S as a ring spectrum.*
- *For R commutative, B(f. g. free R-mod., iso., $\oplus, \otimes_R$) $= K(R)$.*

Symmetric monoidal categories and stable homotopy



Picard groupoids

Fund. Picard grd.

spectra with $\pi_n = 0$ for $n \neq 0, 1$

Sym. mon. cat.

B

May, Segal, Thomason

Spectra

Jibladze–Pirashvili'06
Quang'08

Categorical rings

Ring categories

B

Elmendorf–Mandell'06

Ring spectra

Symmetric monoidal categories and stable homotopy

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honest
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Jibladze–Pirashvili'06
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$$\begin{tikzcd}
\text{Categorical rings} \arrow[d] \arrow[dr, bend left=20, "1\text{-types}"] & \\
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Symmetric monoidal categories and stable homotopy

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\end{tikzcd}$$

How shall we do it?

1. Define a symmetric monoidal ‘replacement’ for the 2-category of Picard groupoids.
3. Construct a ‘lax symmetric monoidal’ 2-functor

put here the ‘replacement’

$$\begin{tikzcd}\boxed{\text{Spectra}} \arrow[r, "\pi_{0,*}", "\text{fundamental Picard groupoid}"'] & \boxed{\text{Picard groupoids}}\end{tikzcd}$$

and more generally, $n \geq 0$,

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All-strict Picard groupoids

A Picard groupoid is strict if $\otimes$ is strictly associative and unital. It is all-strict if $\otimes$ is also strictly commutative.

Any Picard groupoid $\mathbf{P}$ can be *strictified* but not *all-strictified*.

Proposition

The following are equivalent:

- $\mathbf{P}$ *can be all-strictified.*
- $B(\mathbf{P})$ *has trivial Postnikov invariants.*
- *The stable Hopf map* $\eta \in \pi_1(S) \cong \mathbb{Z}/2$ *acts trivially on* $\pi_0(B(\mathbf{P}))$,

$$0 = \pi_0(B(\mathbf{P})) \cdot \eta \subset \pi_1(B(\mathbf{P})).$$

Example

The Picard groupoid $\mathbf{Pic}(X)$ *of line bundles over a scheme* X *can be all-strictified.*

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$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{All-strict}\\ \text{Picard groupoids}\end{array}} \arrow[r, "\simeq"] & \boxed{\begin{array}{c}\text{abelian group}\\ \text{homomorphisms}\end{array}} \\
\mathbf{P} & \{X \to \mathbb{1}\} \xrightarrow{\text{source}} \operatorname{Ob}\mathbf{P}
\end{tikzcd}$$

All-strict Picard groupoids

sym. mon.!

$$\boxed{\begin{array}{c}\text{All-strict}\\ \text{Picard groupoids}\end{array}} \xrightarrow{\qquad\simeq\qquad} \boxed{\begin{array}{c}\text{abelian group}\\ \text{homomorphisms}\end{array}}$$

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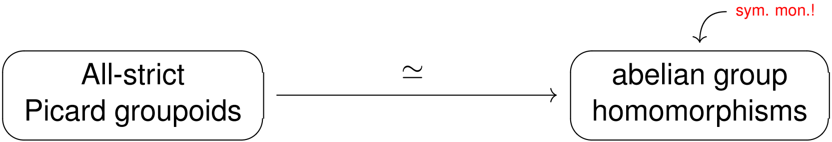

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All-strict Picard groupoids $\xrightarrow{\simeq}$ abelian group homomorphisms

$$\mathbf{P} \qquad\qquad \{X \to \mathbb{1}\} \xrightarrow{\text{source}} \operatorname{Ob}\mathbf{P}$$

$$\begin{tikzcd}
A_1 \arrow[d, "\partial_A"'] \arrow[r, phantom, "\otimes" near end, shift right=6ex] & B_1 \arrow[d, "\partial_B"] \\
A_0 & B_0
\end{tikzcd}$$

All-strict Picard groupoids

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$$\boxed{\text{All-strict Picard groupoids}} \xrightarrow{\quad\simeq\quad} \boxed{\text{abelian group homomorphisms}}$$

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A_1 \arrow[d, "\partial_A"'] & & B_1 \arrow[d, "\partial_B"] & & A_1 \otimes B_1 \arrow[r, "1\otimes\partial_B"] \arrow[d, "\partial_A\otimes 1"'] & A_1 \otimes B_0 \arrow[dd, bend left, "\partial_A\otimes 1"] \\
A_0 & \otimes & B_0 & = & A_0 \otimes B_1 \arrow[dr, bend right, "1\otimes\partial_B"'] & \\
& & & & & A_0 \otimes B_0
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A_0 & & B_0 & & A_0 \otimes B_1 \arrow[r] \arrow[drr, bend right, "1\otimes\partial_B"'] & (A \otimes B)_1 \arrow[dr, "\partial_{A\otimes B}"'] & \\
& & & & & & A_0 \otimes B_0
\end{tikzcd}$$

Crossed bimodules

A monoid in this category is a crossed bimodule C_*,

$$\begin{tikzcd} \underset{C_0\text{-bimod.}}{C_1} \arrow[r, "\partial", "\text{bimod. hom.}"'] & \underset{\text{ring}}{C_0} \end{tikzcd}$$

satisfying

$$c_1 \cdot \partial(c_1') \quad = \quad \partial(c_1) \cdot c_1'.$$

Theorem (Baues-Pirashvili'06)

For R a ring and M an R-bimodule,

$$SH^3(R, M), \quad \textit{Shukla cohomology},$$

classifies crossed bimodules C_* with $h_0 C_* = R$ and $h_1 C_* = M$.

One can similarly consider graded crossed bimodules.

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The all-strict '$\pi_{*,*}$'

Theorem (Baues-M'06)

There are 2*-functors,* $n \geq 0$,

$$\begin{tikzcd} \boxed{\textit{Spectra}} \arrow[r, "\bar{\pi}_{n,*}"] & \boxed{\textit{Abelian group hom.}} \end{tikzcd}$$

$$0 \to \frac{\pi_{n+1}(X)}{\pi_n(X)\cdot\eta} \to \bar{\pi}_{n,1}(X) \xrightarrow{\partial} \bar{\pi}_{n,0}(X) \to \pi_n(X) \to 0.$$

They extend (up to natural quasi-iso.) to a 2*-functor*

$$\begin{tikzcd} \boxed{\textit{Ring spectra}} \arrow[r, "\bar{\pi}_{*,*}"] & \boxed{\textit{Graded crossed bim.}} \end{tikzcd}$$

$$0 \to \frac{\pi_*(R)}{(\eta)}[1] \to \bar{\pi}_{*,1}(R) \xrightarrow{\partial} \bar{\pi}_{*,0}(R) \to \pi_*(R) \to 0.$$

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The functors $\bar{\pi}_{n,*}$ are good enough for spectra X *neglecting the Hopf map*, i.e. such that $\pi_*(X) \cdot \eta = 0$.

Remark

If R is a ring spectrum neglecting η,

$$\{\bar{\pi}_{0,*}(R)\} \in SH^3(\pi_0 R, \pi_1 R) \subset THH^3(\pi_0 R, \pi_1 R)$$

can be identified with the 1st Postnikov invariant of R.

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Remark

If R is a ring spectrum neglecting η,

$$\{\bar{\pi}_{0,*}(R)\} \in SH^3(\pi_0 R, \pi_1 R) \subset THH^3(\pi_0 R, \pi_1 R)$$

can be identified with the 1st Postnikov invariant of R.

Massey products

$$\begin{tikzcd}
0 \arrow[r] & h_1 C_* \arrow[r] & C_1 \arrow[r, "\partial"] & C_0 \arrow[r] & h_0 C_* \arrow[r] & 0
\end{tikzcd}$$

$$x, y, z$$

$$xy = 0$$

$$yz = 0$$

Massey products

$$\begin{tikzcd}
0 \arrow[r] & h_1 C_* \arrow[r] & C_1 \arrow[r, "\partial"] & C_0 \arrow[r] & h_0 C_* \arrow[r] & 0 \\
 & & & \bar{x},\bar{y},\bar{z} \arrow[r, mapsto] & x,y,z & \\
 & & & \bar{x}\bar{y} \arrow[r, mapsto] & xy=0 & \\
 & & & \bar{y}\bar{z} \arrow[r, mapsto] & yz=0 &
\end{tikzcd}$$

Massey products

$$\begin{tikzcd}
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 & & b \arrow[r, mapsto] & \bar{y}\bar{z} \arrow[r, mapsto] & yz=0 &
\end{tikzcd}$$

Massey products

$$\begin{tikzcd}
0 \arrow[r] & h_1 C_* \arrow[r] & C_1 \arrow[r, "\partial"] & C_0 \arrow[r] & h_0 C_* \arrow[r] & 0 \\
& -a\bar{z}+\bar{x}b & & \bar{x},\bar{y},\bar{z} \arrow[r, mapsto] & x,y,z & \\
& \in \langle x,y,z\rangle & a \arrow[r, mapsto] & \bar{x}\bar{y} \arrow[r, mapsto] & xy=0 & \\
& \text{Massey product} & b \arrow[r, mapsto] & \bar{y}\bar{z} \arrow[r, mapsto] & yz=0 &
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Massey products

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0 \arrow[r] & h_1 C_* \arrow[r] & C_1 \arrow[r, "\partial"] & C_0 \arrow[r] & h_0 C_* \arrow[r] & 0 \\
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& \text{Massey product} & b \arrow[r, mapsto] & \bar{y}\bar{z} \arrow[r, mapsto] & yz=0 &
\end{tikzcd}$$

For R a ring spectrum neglecting η and $C_* = \bar{\pi}_{*,*}(R)$ if we have $x, y, z \in \pi_*(R)$ then

$$\langle x, y, z\rangle \quad \subset \quad \pi_{|x|+|y|+|z|+1}(R)$$

is the Toda bracket.

The all-strict '$\pi_{*,*}$'

Example

The ring spectrum $H\mathbb{F}_2 \wedge H\mathbb{F}_2$ neglects the Hopf map, and $\bar{\pi}_{*,*}(H\mathbb{F}_2 \wedge H\mathbb{F}_2)$ is (quasi-iso. to) the dual secondary Steenrod algebra.

The secondary Steenrod algebra $\mathcal{B}$ is a graded crossed bimodule

$$0 \to \mathcal{A}[-1] \to \mathcal{B}_1 \xrightarrow{\partial} \mathcal{B}_0 \to \underbrace{\mathcal{A}}_{\text{Steenrod algebra}} \to 0,$$

actually a 2-Hopf algebra, computed by Baues'06. The cohomology of $\mathcal{B}$ leads to a direct computation of E_3 of Adams SS [Baues-Jibladze'06].

The commutative case

A **commutative crossed bimodule** C_*,

$$\begin{tikzcd} C_1 \arrow[r, "\partial", "\text{module hom.}"'] & C_0 \\ C_0\text{-mod.} & \text{comm. ring} \end{tikzcd}$$

They are not enough! We need **graded shc crossed bimodules**,

$$\begin{tikzcd} C_{*,1} \arrow[r, "\partial", "\text{graded crossed bim.}"'] & C_{*,0} \end{tikzcd}$$

$$\smile_1 \colon C_{*,0} \otimes C_{*,0} \longrightarrow C_{*,1}$$

$$\begin{aligned} \partial(c_0 \smile_1 c_0') &= c_0 c_0' - (-1)^{|c_0||c_0'|} c_0' c_0, \\ \partial(c_1) \smile_1 c_0 &= c_1 c_0 - (-1)^{|c_1||c_0|} c_0 c_1, \\ 0 &= c_0 \smile_1 c_0' + (-1)^{|c_0||c_0'|} c_0 \smile_1 c_0', \\ (c_0 c_0') \smile_1 c_0'' &= (-1)^{|c_0'||c_0''|}(c_0 \smile_1 c_0'') c_0' + c_0 (c_0' \smile_1 c_0''). \end{aligned}$$

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$$\begin{aligned}
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(c_0 c_0') \smile_1 c_0'' &= (-1)^{|c_0'||c_0''|}(c_0 \smile_1 c_0'')c_0' + c_0(c_0' \smile_1 c_0'').
\end{aligned}$$

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$$C_{*,1} \xrightarrow[\text{graded crossed bim.}]{\partial} C_{*,0}$$

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\partial(c_1) \smile_1 c_0 &= c_1 c_0 - (-1)^{|c_1||c_0|} c_0 c_1, \\
0 &= c_0 \smile_1 c_0' + (-1)^{|c_0||c_0'|} c_0 \smile_1 c_0', \\
(c_0 c_0') \smile_1 c_0'' &= (-1)^{|c_0'||c_0''|}(c_0 \smile_1 c_0'') c_0' + c_0 (c_0' \smile_1 c_0'').
\end{aligned}$$

The commutative case

A commutative crossed bimodule C_*,

$$\begin{tikzcd} \underset{C_0\text{-mod.}}{C_1} \arrow[r, "\partial", "\text{module hom.}"'] & \underset{\text{comm. ring}}{C_0} \end{tikzcd}$$

They are not enough! We need graded shc crossed bimodules,

$$\begin{tikzcd} C_{*,1} \arrow[r, "\partial", "\text{graded crossed bim.}"'] & C_{*,0} \arrow[r] & \underset{\text{graded comm. ring}}{h_0 C_*} \arrow[r] & 0 \end{tikzcd}$$

$$\smile_1 : C_{*,0} \otimes C_{*,0} \longrightarrow C_{*,1}$$

$$\begin{aligned}
\partial(c_0 \smile_1 c_0') &= c_0 c_0' - (-1)^{|c_0||c_0'|} c_0' c_0, \\
\partial(c_1) \smile_1 c_0 &= c_1 c_0 - (-1)^{|c_1||c_0|} c_0 c_1, \\
0 &= c_0 \smile_1 c_0' + (-1)^{|c_0||c_0'|} c_0 \smile_1 c_0', \\
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\end{aligned}$$

The commutative case

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$$\begin{tikzcd} C_1 \arrow[r, "\partial", "\text{module hom.}"'] & C_0 \\ \text{C_0-mod.} & \text{comm. ring} \end{tikzcd}$$

They are not enough! We need graded shc crossed bimodules,

$$\begin{tikzcd} 0 \arrow[r] & h_1 C_* \arrow[r] & C_{*,1} \arrow[r, "\partial", "\text{graded crossed bim.}"'] & C_{*,0} \arrow[r] & h_0 C_* \arrow[r] & 0 \\ & \text{graded $h_0 C_*$-mod.} & & & \text{graded comm. ring} & \end{tikzcd}$$

$$\smile_1 : C_{*,0} \otimes C_{*,0} \longrightarrow C_{*,1}$$

$$\begin{aligned} \partial(c_0 \smile_1 c_0') &= c_0 c_0' - (-1)^{|c_0||c_0'|} c_0' c_0, \\ \partial(c_1) \smile_1 c_0 &= c_1 c_0 - (-1)^{|c_1||c_0|} c_0 c_1, \\ 0 &= c_0 \smile_1 c_0' + (-1)^{|c_0||c_0'|} c_0 \smile_1 c_0', \\ (c_0 c_0') \smile_1 c_0'' &= (-1)^{|c_0'||c_0''|} (c_0 \smile_1 c_0'') c_0' + c_0 (c_0' \smile_1 c_0''). \end{aligned}$$

The commutative case

Theorem (Baues-M'06)

The 2-functor $\bar{\pi}_{*,*}$ above extends (up to natural quasi-iso.) to a 2-functor

$$\boxed{\text{Comm. ring spectra}} \xrightarrow{\quad\bar{\pi}_{*,*}\quad} \boxed{\text{Graded shc crossed bim.}}$$

$$0 \to \frac{\pi_*(R)}{(\eta)}[1] \to \bar{\pi}_{*,1}(R) \xrightarrow{\partial} \bar{\pi}_{*,0}(R) \to \pi_*(R) \to 0,$$

$$\smile_1 : \bar{\pi}_{*,0}(R) \otimes \bar{\pi}_{*,0}(R) \longrightarrow \bar{\pi}_{*,1}(R).$$

A commutative example

For the 3-local sphere commutative ring spectrum $S_{(3)}$,

n	0	2	3	6	7	9	10	11	12	13
π_n	$\overset{1}{\mathbb{Z}_{(3)}}$	0	$\overset{\alpha_1}{\mathbb{Z}/3}$	0	$\overset{\alpha_2}{\mathbb{Z}/3}$	0	$\overset{\beta_1}{\mathbb{Z}/3}$	$\overset{\alpha'_3}{\mathbb{Z}/9}$	0	$\overset{\alpha_1\cdot\beta_1}{\mathbb{Z}/3}$
$\pi_{n,*}$	$\begin{array}{c} \overset{1}{\mathbb{Z}_{(3)}} \\ \uparrow \\ 0 \end{array}$	$\begin{array}{c} 0 \\ \uparrow \\ \underset{\alpha_1[1]}{\mathbb{Z}/3} \end{array}$	$\begin{array}{c} \overset{a_1}{\mathbb{Z}_{(3)}} \\ {\scriptstyle 3}\uparrow \\ \underset{\bar{a}_1}{\mathbb{Z}_{(3)}} \end{array}$	$\begin{array}{c} \overset{a_1^2}{\mathbb{Z}_{(3)}} \\ \uparrow {\scriptstyle (1\ 0)} \\ \underset{\bar{\bar{a}}_1}{\mathbb{Z}_{(3)}} \oplus \underset{\alpha_2[1]}{\mathbb{Z}/3} \end{array}$	$\begin{array}{c} \overset{a_2}{\mathbb{Z}_{(3)}} \\ {\scriptstyle 3}\uparrow \\ \underset{\bar{a}_2}{\mathbb{Z}_{(3)}} \end{array}$	$\begin{array}{c} \overset{a_1^3}{\mathbb{Z}_{(3)}} \\ \uparrow {\scriptstyle (1\ 0)} \\ \underset{a_1\bar{\bar{a}}_1}{\mathbb{Z}_{(3)}} \oplus \underset{\beta_1[1]}{\mathbb{Z}/3} \end{array}$	$\begin{array}{c} \overset{\substack{a_1a_2 \\ a_2a_1 \\ b_1}}{\mathbb{Z}_{(3)}^3} \\ {\scriptstyle \partial}\uparrow \\ \underset{\substack{\bar{a}_{1,2} \\ \bar{a}_{2,1} \\ \bar{b}_1}}{\mathbb{Z}_{(3)}^3} \oplus \underset{\alpha'_3[1]}{\mathbb{Z}/9} \end{array}$	$\begin{array}{c} \overset{a'_3}{\mathbb{Z}_{(3)}} \\ {\scriptstyle 9}\uparrow \\ \underset{\bar{a}'_3}{\mathbb{Z}_{(3)}} \end{array}$	$\begin{array}{c} \overset{a_1^4}{\mathbb{Z}_{(3)}} \\ \uparrow {\scriptstyle (1\ 0)} \\ \underset{a_1^2\bar{\bar{a}}_1}{\mathbb{Z}_{(3)}} \oplus \underset{\alpha_1[1]b_1}{\mathbb{Z}/3} \end{array}$	?

$$
\begin{aligned}
a_1\bar{a}_2 &= 3\bar{a}_{1,2}+x\alpha'_3[1],\\
\bar{a}_1a_2 &= 3\bar{a}_{1,2}+(x-3)\alpha'_3[1],\\
a_1\smile_1 a_1 &= 2\bar{\bar{a}}_1,\\
a_1\smile_1 a_2 &= \bar{a}_{1,2}+\bar{a}_{2,1},
\end{aligned}
\qquad
\partial = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}
$$

where $x \in \mathbb{Z}/9$ is an unknown constant!

Don't neglect η!

stable quadratic modules $\xrightarrow{\simeq}$ Picard groupoids

▸ sqm ▸ jump

Don’t neglect η!

$$\begin{tikzcd}\boxed{\text{stable quadratic modules}} \arrow[r, "\simeq"] & \boxed{\text{Picard groupoids}}\end{tikzcd}$$

$$\begin{tikzcd}C_0^{ab}\otimes C_0^{ab} \arrow[d, "\langle\cdot,\cdot\rangle"'] & \\ C_1 \arrow[r, "\partial"'] & C_0\end{tikzcd}$$

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C_0^{ab} \otimes C_0^{ab} \arrow[d, "\langle\cdot,\cdot\rangle"'] & \\
C_1 \arrow[r, "\partial"'] & C_0
\end{tikzcd}$$

$$\begin{aligned}
\partial\langle c_0, c_0'\rangle &= [c_0', c_0]\\
\langle\partial(c_1), \partial(c_1')\rangle &= [c_1', c_1]\\
\langle c_0, c_0'\rangle &= -\langle c_0', c_0\rangle
\end{aligned}$$

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\langle c_0, c_0'\rangle &= -\langle c_0', c_0\rangle
\end{aligned}$$

$\mathbf{P}C_*$

Object set: C_0

Maps: $c_0 + \partial(c_1) \xrightarrow{c_1} c_0$

$$\otimes = +$$

$$c_0 + c_0' \xrightarrow{\langle c_0, c_0'\rangle} c_0' + c_0$$

▸ sqm ▸ jump

Don’t neglect η!

not symmetric monoidal

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$$\mathbf{P}C_*$$

Object set: C_0

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▸ sqm ▸ jump

Don't neglect η!

symmetric monoidal!

$$\begin{tikzcd} \boxed{\text{quadratic pair modules}} \arrow[r, "\simeq"] & \boxed{\text{Picard groupoids}} \end{tikzcd}$$

$$\begin{tikzcd} C_0^{ab} \otimes C_0^{ab} \arrow[d, "\langle\cdot,\cdot\rangle"'] & \\ C_1 \arrow[r, "\partial"'] & C_0 \arrow[ul, bend right, "H"'] \end{tikzcd}$$

$$\begin{aligned} \partial\langle c_0, c_0'\rangle &= [c_0', c_0] \\ \langle\partial(c_1), \partial(c_1')\rangle &= [c_1', c_1] \\ \langle c_0, c_0'\rangle &= -\langle c_0', c_0\rangle \\ H(c_0 + c_0') &= H(c_0) + H(c_0') + c_0' \otimes c_0 \end{aligned}$$

$\mathbf{P}C_*$

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▸ sqm ▸ jump

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symmetric monoidal! Baues-M'06, Baues-Jibladze-Pirashvili'08

$$\boxed{\text{quadratic pair modules}} \xrightarrow{\simeq} \boxed{\text{Picard groupoids}}$$

$$\begin{tikzcd}
C_0^{ab}\otimes C_0^{ab} \arrow[d, "\langle\cdot,\cdot\rangle"'] & \\
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\end{tikzcd}$$

$$\mathbf{P}C_*$$

Object set: C_0

Maps: $c_0+\partial(c_1)\xrightarrow{c_1}c_0$

$$\otimes = +$$

$$c_0+c_0' \xrightarrow{\langle c_0,c_0'\rangle} c_0'+c_0$$

$$\begin{aligned}
\partial\langle c_0,c_0'\rangle &= [c_0',c_0]\\
\langle\partial(c_1),\partial(c_1')\rangle &= [c_1',c_1]\\
\langle c_0,c_0'\rangle &= -\langle c_0',c_0\rangle\\
H(c_0+c_0') &= H(c_0)+H(c_0')+c_0'\otimes c_0
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\end{aligned}$$

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$$c_0 + c_0' \xrightarrow{\langle c_0, c_0' \rangle} c_0' + c_0$$

▸ sqm ▸ jump

We may want to assume that Ker $H \subset C_0 \to h_0 C_*$ is surjective.

The 2-category of stable quadratic modules

A morphism $f\colon C_* \to D_*$ is a chain morphism with $\langle f_0, f_0\rangle = f_1\langle\cdot,\cdot\rangle$ back

$$\begin{tikzcd}
C_1 \arrow[r, "f_1"] \arrow[d, "\partial"'] & D_1 \arrow[d, "\partial"] \\
C_0 \arrow[r, "f_0"'] & D_0
\end{tikzcd}$$

The 2-category of stable quadratic modules

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$$\begin{tikzcd}
C_1 \arrow[r, "f_1"] \arrow[r, bend left=60, "g_1"] \arrow[d, "\partial"'] & D_1 \arrow[d, "\partial"] \\
C_0 \arrow[r, "f_0"'] \arrow[r, bend right=60, "g_0"'] & D_0
\end{tikzcd}$$

The 2-category of stable quadratic modules

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back

$$\begin{tikzcd}
C_1 \arrow[r, "f_1"] \arrow[r, bend left=60, "g_1"] \arrow[d, "\partial"'] & D_1 \arrow[d, "\partial"] \\
C_0 \arrow[r, "f_0"'] \arrow[r, bend right=60, "g_0"'] \arrow[ur, "\alpha"] & D_0
\end{tikzcd}$$

A track, homotopy or 2-morphism $\alpha\colon f \Rightarrow g$ is a map such that

$$\begin{aligned}
\partial\alpha(c_0) &= -g_0(c_0) + f_0(c_0),\\
\alpha\partial(c_1) &= -g_1(c_1) + f_1(c_1),\\
\alpha(c_0 + c_0') &= \alpha(c_0) + \alpha(c_0') + \langle g_0(c_0'), \partial\alpha(c_0)\rangle.
\end{aligned}$$

Quadratic pair modules

Example

Let C_*

$$\begin{tikzcd}
C_0^{ab} \otimes C_0^{ab} \arrow[d, "{\langle\cdot,\cdot\rangle}"'] & \\
C_1 \arrow[r, "\partial"'] & C_0 = \langle E\rangle^{nil}
\end{tikzcd}$$

be a stable quadratic module such that $C_0 = \langle E\rangle^{nil}$ *is freely generated by a set* E *as a group of nilpotency class* 2.

There exists a unique map H *satisfying* $H(e) = 0$ *for any* $e \in E$ *which turns* C_* *into a quadratic pair module.*

Quadratic pair modules

Example

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There exists a unique map H *satisfying* $H(e) = 0$ *for any* $e \in E$ *which turns* C_* *into a quadratic pair module.*

Quadratic pair algebras

A quadratic pair algebra is,

$$\begin{tikzcd}
\otimes^2 C_0^{ab} \arrow[d, "P=\langle\cdot,\cdot\rangle"'] & \\
C_1 \arrow[r, "\partial", "\text{bimod. hom.}"'] & C_0 \arrow[ul, bend right, "H"'] \\
C_0\text{-bimod.} & \text{monoid}
\end{tikzcd}$$

$$\partial(c_1)c_1' \quad = \quad c_1\partial(c_1'),$$

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\text{C_0-bimod.} & \text{monoid}
\end{tikzcd}$$

$$\begin{aligned}
\partial(c_1)c_1' &= c_1\partial(c_1'),\\
c_i(c_j'+c_j'') &= c_ic_j'+c_ic_j'', \quad 0\leq i,j,i+j\leq 1,
\end{aligned}$$

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$$\begin{aligned}
\partial(c_1)c_1' &= c_1\partial(c_1'),\\
c_i(c_j'+c_j'') &= c_ic_j'+c_ic_j'', \quad 0\leq i,j,i+j\leq 1,\\
(c_0+c_0')c_0'' &= c_0c_0''+c_0'c_0''+\partial P((c_0\otimes c_0')H(c_0'')),\\
(c_0+c_0')c_1 &= c_0c_1+c_0'c_1+P((c_0\otimes c_0')H\partial(c_1)),\\
(c_1+c_1')c_0 &= c_1c_0+c_1'c_0+P((\partial(c_1)\otimes\partial(c_1'))H(c_0)),
\end{aligned}$$

Quadratic pair algebras

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$$\begin{tikzcd}
\otimes^2 C_0^{ab} \arrow[d, "P=\langle\cdot,\cdot\rangle"'] & \\
C_1 \arrow[r, "\partial", "\text{bimod. hom.}"'] & C_0 \arrow[ul, bend right, "H"'] \\
\text{C_0-bimod.} & \text{monoid}
\end{tikzcd}$$

$$\begin{aligned}
\partial(c_1)c_1' &= c_1\partial(c_1'),\\
c_i(c_j'+c_j'') &= c_ic_j'+c_ic_j'', \quad 0\leq i,j,i+j\leq 1,\\
(c_0+c_0')c_0'' &= c_0c_0''+c_0'c_0''+\partial P((c_0\otimes c_0')H(c_0'')),\\
(c_0+c_0')c_1 &= c_0c_1+c_0'c_1+P((c_0\otimes c_0')H\partial(c_1)),\\
(c_1+c_1')c_0 &= c_1c_0+c_1'c_0+P((\partial(c_1)\otimes\partial(c_1'))H(c_0)),\\
H(1) &= 0,\\
H(c_0c_0') &= (c_0\otimes c_0)H(c_0')+H(c_0)(c_0'\otimes c_0'+H\partial PH(c_0')-2H(c_0')),
\end{aligned}$$

Quadratic pair algebras

A quadratic pair algebra is,

$$\begin{tikzcd}
\otimes^2 C_0^{ab} \arrow[d, "P=\langle\cdot,\cdot\rangle"'] & \\
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$$\begin{aligned}
\partial(c_1)c'_1 &= c_1\partial(c'_1),\\
c_i(c'_j+c''_j) &= c_ic'_j+c_ic''_j, \quad 0\leq i,j,i+j\leq 1,\\
(c_0+c'_0)c''_0 &= c_0c''_0+c'_0c''_0+\partial P((c_0\otimes c'_0)H(c''_0)),\\
(c_0+c'_0)c_1 &= c_0c_1+c'_0c_1+P((c_0\otimes c'_0)H\partial(c_1)),\\
(c_1+c'_1)c_0 &= c_1c_0+c'_1c_0+P((\partial(c_1)\otimes\partial(c'_1))H(c_0)),\\
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H(c_0c'_0) &= (c_0\otimes c_0)H(c'_0)+H(c_0)(c'_0\otimes c'_0+H\partial PH(c'_0)-2H(c'_0)),\\
c_0P(c_1)c'_0 &= P((c_0\otimes c_0)c_1(c'_0\otimes c'_0+H\partial PH(c'_0)-2H(c'_0))).
\end{aligned}$$

Quadratic pair algebras

A quadratic pair algebra is,

$$\begin{tikzcd}
\otimes^2 C_0^{ab} \arrow[d, "P=\langle\cdot,\cdot\rangle"'] & & & \\
C_1 \arrow[r, "\partial", "\text{bimod. hom.}"'] & C_0 \arrow[r] \arrow[lu, bend right, "H"'] & h_0 C_* \arrow[r] & 0 \\
\text{C_0-bimod.} & \text{monoid} & \text{ring} &
\end{tikzcd}$$

$$\begin{aligned}
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 & h_0 C_*\text{-bimod.} & C_0\text{-bimod.} & \text{monoid} & \text{ring} &
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\end{aligned}$$

Quadratic pair algebras

Theorem (Baues-Jibladze-Pirashvili'06)

For R a ring and M an R-bimodule,

$$THH^3(R, M), \quad \textit{topological Hochschild cohomology},$$

classifies quadratic pair algebras C_* with $h_0 C_* = R$ and $h_1 C_* = M$.

The right $\pi_{n,*}$

Theorem (Baues-M'06)

There are 2-functors, $n \geq 0$,

$$\begin{tikzcd} \boxed{\textit{Spectra}} \arrow[r, "\pi_{n,*}"] & \boxed{\textit{quadratic pair mod.}} \end{tikzcd}$$

$$0 \to \pi_{n+1}(X) \to \pi_{n,1}(X) \xrightarrow{\partial} \pi_{n,0}(X) \to \pi_n(X) \to 0.$$

They extend (up to natural quasi-iso.) to a 2-functor

$$\begin{tikzcd} \boxed{\textit{Ring spectra}} \arrow[r, "\pi_{*,*}"] & \boxed{\textit{Graded quadratic pair alg.}} \end{tikzcd}$$

$$0 \to \pi_*(R)[1] \to \pi_{*,1}(R) \xrightarrow{\partial} \pi_{*,0}(R) \to \pi_*(R) \to 0.$$

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Quadratic pair algebras

Remark

If R is a ring spectrum,

$$\{\pi_{0,*}(R)\} \in THH^3(\pi_0 R, \pi_1 R)$$

can be identified with the 1st Postnikov invariant of R.

Example

The quadratic pair algebra $\pi_{0,*}(S)$ *is equivalent to*

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, two heads, "\langle \cdot , \cdot \rangle"'] & \\
\mathbb{Z}/2 \arrow[r, "\partial=0"'] & \mathbb{Z} \arrow[ul, bend right, "H(n)=\frac{n(n-1)}{2}"']
\end{tikzcd}$$

which generates $THH^3(\mathbb{Z}, \mathbb{Z}/2) \cong \mathbb{Z}/2$.

Quadratic pair algebras

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Quadratic pair algebras

Example

If F is a field, the quadratic pair algebra $\pi_{0,*}(K(F))$ is equivalent to

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, "1\mapsto -1"'] & \\
F^\times \arrow[r, "\partial=0"'] & \mathbb{Z} \arrow[ul, bend right, "H(n)=\frac{n(n-1)}{2}"']
\end{tikzcd}$$

which is non-trivial in $THH^3(\mathbb{Z}, F^\times) \cong {}_2(F^\times)$ unless char $F = 2$.

Example

Let $\mathbf{C}$ be a monoidal exact or Waldhausen category, $\pi_{0,*}(K(\mathbf{C}))$ in the next talk!

Quadratic pair algebras

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Example

Let **C** *be a monoidal exact or Waldhausen category, $\pi_{0,*}(K(\mathbf{C}))$ in the next talk!*

$\pi_{n,*}$ for spaces

Let X be a pointed space, $n > 2$,

$$\begin{tikzcd}
\otimes^2\pi_{n,0}(X)^{ab} \arrow[d, "\langle\cdot,\cdot\rangle"'] & \\
\pi_{n,1}(X) \arrow[r, "\partial"'] & \pi_{n,0}(X) = \langle\{S^n \to X\}\rangle^{nil} \arrow[ul, bend right, "H"']
\end{tikzcd}$$

An element $[f,F] \in \pi_{n,1}(X)$ is *represented* by $f\colon S^1 \to \bigvee_{S^n\to X} S^1$ and

$$\begin{tikzcd}
S^n \arrow[r, "S^{n-1}\wedge f"'] \arrow[rr, bend left=50, "0"{name=U}] & \bigvee_{S^n\to X} S^n \arrow[r, "ev"'] \arrow[to=U, Rightarrow, "F"'] & X.
\end{tikzcd}$$

$$\begin{array}{ccccccc}
\mathbb{Z} & = & \pi_1(S^1) \xrightarrow{\pi_1(f)} \pi_1(\bigvee_{S^n\to X} S^n) & = & \langle\{S^n \to X\}\rangle & \twoheadrightarrow & \langle\{S^n \to X\}\rangle^{nil} \\
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\end{tikzcd}
\quad \Uparrow F$$

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\end{array}$$

The Hopf invariant for tracks

For $n = 2$,

$$\begin{tikzcd} S^2 \arrow[r, bend left, "S^1 \wedge f"{name=U}] \arrow[r, bend right, "S^1 \wedge g"'{name=D}] & S^2 \vee \overset{k}{\cdots} \vee S^2 \arrow[Rightarrow, from=U, to=D, "T"] \end{tikzcd}$$

$$\color{red}{\mathrm{Hopf}(T)} \in \otimes^2 \mathbb{Z}^k \cong \mathbb{Z}^{k^2}$$

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$$[0,1]_+ \wedge S^2 \xrightarrow{T} S^2\vee \overset{k}{\cdots} \vee S^2$$

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\end{tikzcd}$$

$$\mathrm{Hopf}(T) \in \otimes^2\mathbb{Z}^k \cong \mathbb{Z}^{k^2}$$

$$[0,1]_+ \wedge S^1 \xrightarrow{ad(T)} \Omega(S^2\vee \overset{k}{\cdots} \vee S^2)$$

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$\mathrm{Hopf}(T) \in \otimes^2 \mathbb{Z}^k \cong \mathbb{Z}^{k^2}$

$$\begin{tikzcd} {[0,1]_+ \wedge S^1} \arrow[r, "ad(T)"] & \Omega(S^2 \vee \overset{k}{\cdots} \vee S^2) \\ \{0,1\}_+ \wedge S^1 \arrow[u, "\text{incl.}"] & \end{tikzcd}$$

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Hopf$(T) \in \otimes^2\mathbb{Z}^k \cong \mathbb{Z}^{k^2}$

$$H_2([0,1]_+\wedge S^1,\{0,1\}_+\wedge S^1) \xrightarrow{ad(T)_*} H_2(\Omega(S^2\vee\overset{k}{\cdots}\vee S^2),S^1\vee\overset{k}{\cdots}\vee S^1)$$

The Hopf invariant for tracks

For $n = 2$,

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The Hopf invariant for tracks

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\end{tikzcd}$$

$$\mathrm{Hopf}(T) \in \bigotimes{}^2\mathbb{Z}^k \cong \mathbb{Z}^{k^2}$$

$$\begin{tikzcd}[column sep=huge]
H_2([0,1]_+\wedge S^1,\{0,1\}_+\wedge S^1) \arrow[r, "ad(T)_*"] & H_2(\Omega(S^2\vee\overset{k}{\cdots}\vee S^2),S^1\vee\overset{k}{\cdots}\vee S^1) \\
\mathbb{Z} \arrow[u, "\cong"] & \bigotimes{}^2\,\mathbb{Z}^k \arrow[u, "\text{Pontryagin}", "\cong"']
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\mathbb{Z} \arrow[u, "\cong"] & \otimes^2\mathbb{Z}^k \arrow[u, "\text{Pontryagin}", "\cong"'] \\
1 &
\end{tikzcd}$$

The Hopf invariant for tracks

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$$\begin{tikzcd} H_2([0,1]_+\wedge S^1,\{0,1\}_+\wedge S^1) \arrow[r, "ad(T)_*"] & H_2(\Omega(S^2\vee\overset{k}{\cdots}\vee S^2),S^1\vee\overset{k}{\cdots}\vee S^1) \\ \mathbb{Z} \arrow[u, "\cong"] & \otimes^2\mathbb{Z}^k \arrow[u, "\text{Pontryagin}", "\cong"'] \\ 1 \arrow[r, mapsto] & \text{Hopf}(T) \end{tikzcd}$$

For $n > 2$ we need the reduced tensor square $\hat{\otimes}^2 A = \frac{A\otimes A}{(a\otimes b + b\otimes a)}$,

$$\text{Hopf}(T) \in \hat{\otimes}^2\mathbb{Z}^k \cong \mathbb{Z}^{\frac{k(k-1)}{2}} \oplus (\mathbb{Z}/2)^k.$$

$\pi_{n,*}$ for spaces

Two elements $[f, F] = [g, G] \in \pi_{n,1}(X)$ coincide iff

$$\begin{tikzcd}[column sep=huge]
S^n \arrow[r, bend left=15, "S^{n-1}\wedge f"] \arrow[r, bend right=15, "S^{n-1}\wedge g"'] \arrow[rr, bend left=60, "0", ""{name=U, below}] \arrow[rr, bend right=60, "0"', ""{name=D, above}] & \displaystyle\bigvee_{S^n \to X} S^n \arrow[r, "ev"] \arrow[to=U, Rightarrow, "F"'] \arrow[to=D, Rightarrow, "G"] & X
\end{tikzcd}$$

there exists a track T such that $\mathrm{Hopf}(T) = 0$ and the pasting of the diagram is the identity track.

We still have to define $\langle \cdot, \cdot \rangle \colon \otimes^2 \pi_{n,0}(X)^{ab} \longrightarrow \pi_{n,1}(X)$.

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Two elements $[f,F]=[g,G]\in\pi_{n,1}(X)$ coincide iff

$$\begin{tikzcd}
S^n \arrow[r, bend left, "S^{n-1}\wedge f"] \arrow[r, bend right, "S^{n-1}\wedge g"'] \arrow[r, phantom, "\Downarrow T"] \arrow[rr, bend left=60, "0"] \arrow[rr, bend right=60, "0"'] & \bigvee_{S^n\to X} S^n \arrow[r, "ev"] \arrow[r, phantom, shift left=8, "\Uparrow F"] \arrow[r, phantom, shift right=8, "\Downarrow G"] & X
\end{tikzcd}$$

there exists a track T such that $\mathrm{Hopf}(T)=0$ and the pasting of the diagram is the identity track.

We still have to define $\langle\cdot,\cdot\rangle\colon \otimes^2\pi_{n,0}(X)^{ab}\longrightarrow\pi_{n,1}(X)$.

$\pi_{n,*}$ for spaces

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there exists a track T such that $\mathrm{Hopf}(T) = 0$ and the pasting of the diagram is the identity track.

We still have to define $\langle\cdot,\cdot\rangle\colon \otimes^2 \pi_{n,0}(X)^{ab} \longrightarrow \pi_{n,1}(X)$.

$\pi_{n,*}$ for spaces

Let $c\colon S^1 \to S^1 \vee S^1$ be a map such that $\pi_1(c)\colon \mathbb{Z} \to \langle i_1, i_1 \rangle\colon 1 \mapsto [i_2, i_1]$.

For $n > 2$, there exists a unique track C

$$\begin{tikzcd}
S^n \arrow[r, bend left=50, "0", ""{name=U, below}] \arrow[r, bend right=50, "S^{n-1}\wedge c"', ""{name=D}] & S^n \vee S^n
\arrow[Rightarrow, from=D, to=U, "C"]
\end{tikzcd}$$

with $\mathrm{Hopf}(C) = i_1 \otimes i_2 \in \hat{\otimes}^2 \langle i_1, i_2 \rangle^{ab}$.

Given $e, e'\colon S^n \to X$ in $\pi_{n,0}(X)$, the element $\langle e, e' \rangle \in \pi_{n,1}(X)$ is

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S^n \arrow[r, "S^{n-1}\wedge c"', ""{name=D}] \arrow[r, bend left=40, "0", ""{name=U, below}] & S^n \vee S^n \arrow[r, phantom, "\overset{e,e'}{\subset}"] & \bigvee\limits_{S^n \to X} S^n \arrow[r, "ev"'] & X
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\arrow[Rightarrow, from=D, to=U, "C"]
\end{tikzcd}$$

$\pi_{n,*}$ for spaces

Example

For $n > 2$, $\pi_{n,*}(S^n \vee \overset{k}{\cdots} \vee S^n)$ is quasi-isomorphic to

$$\begin{tikzcd}
\otimes^2\mathbb{Z}^k \arrow[d, two heads, "\langle\cdot,\cdot\rangle"'] & \\
\hat{\otimes}^2\mathbb{Z}^k \arrow[r, "\textit{anticommutator}", "\partial"'] & \langle i_1, \ldots, i_k\rangle^{nil} \arrow[ul, bend right, "H"']
\end{tikzcd}$$

An element $a \in \hat{\otimes}^2\mathbb{Z}^k$ can be identified with

$$\begin{tikzcd}
S^n \arrow[r, bend left=50, "0"{name=U}] \arrow[r, "S^{n-1}\wedge f"'{name=D}] & S^n \vee \overset{k}{\cdots} \vee S^n \arrow[r, phantom, "\overset{i_j}{\subset}"] & \bigvee\limits_{S^n \to X} S^n \arrow[r, "ev"'] & X = S^n \vee \overset{k}{\cdots} \vee S^n
\arrow[Rightarrow, from=D, to=U, "F"']
\end{tikzcd}$$

where F is any track with $\mathrm{Hopf}(F) = a$.

$\pi_{n,*}$ for spaces

Example

For $n > 2$, $\pi_{n,*}(S^n\vee\overset{k}{\cdots}\vee S^n)$ is quasi-isomorphic to

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\end{tikzcd}$$

An element $a \in \hat{\otimes}^2\mathbb{Z}^k$ can be identified with

$$\begin{tikzcd}
S^n \arrow[r, "S^{n-1}\wedge f"', ""{name=A}] \arrow[r, bend left=50, "0", ""{name=B}] & S^n\vee\overset{k}{\cdots}\vee S^n \arrow[r, hook, "i_j"] & \bigvee_{S^n\to X} S^n \arrow[r, "ev"'] & X = S^n\vee\overset{k}{\cdots}\vee S^n
\arrow[Rightarrow, from=A, to=B, "F"']
\end{tikzcd}$$

where F is any track with $\mathrm{Hopf}(F) = a$.

The End

Thanks for your attention!