Exotic triangulated categories

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(joint work with S. Schwede and N. Strickland)

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- Exhibiting triangulated categories without models.

Models

Let $\mathcal{M}$ be a model category with a zero object 0. Suspensions are defined as

$$\Sigma X \quad = \quad \text{homotopy cofiber of } X \to 0.$$

If the functor

$$\Sigma\colon \operatorname{Ho}\mathcal{M} \longrightarrow \operatorname{Ho}\mathcal{M}$$

is an equivalence then $\mathcal{M}$ is a stable model category.

Example

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The axioms of a triangulated category encode the fundamental properties of cofiber sequences in $\operatorname{Ho}\mathcal{M}$,

$$A \xrightarrow{f} B \xrightarrow{i} \mathrm{Cof}(f) \xrightarrow{q} \Sigma A.$$

Let $\mathcal{T}$ be an additive category and $\Sigma\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ a self-equivalence.

A candidate triangle is a sequence

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such that

$$\begin{aligned} if &= 0, \\ qi &= 0, \\ (\Sigma f)q &= 0. \end{aligned}$$

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Triangulated categories

A triangulated category $(\mathcal{T}, \Sigma, \mathcal{E})$ is a pair $(\mathcal{T}, \Sigma)$ as above together with a replete family $\mathcal{E}$ of candidate triangles, called exact triangles, such that

- The trivial triangle $A \to A \to 0 \to \Sigma A$ is exact,
- $A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{q} \Sigma A$ is exact $\Leftrightarrow$ the translate $B \xrightarrow{-i} C \xrightarrow{-q} \Sigma A \xrightarrow{-\Sigma f} \Sigma B$ is exact,
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- $$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "i"] \arrow[d, "\beta"'] & C \arrow[r, "q"] \arrow[d, "\exists"', "\gamma"] & \Sigma A \arrow[d, "\Sigma\alpha"] & \text{exact} \\ A' \arrow[r, "f'"'] & B' \arrow[r, "i'"'] & C' \arrow[r, "q'"'] & \Sigma A' & \text{exact} \end{tikzcd}$$

  in such a way that the mapping cone of (α, β, γ)

  $$B \oplus A' \xrightarrow{\begin{pmatrix} -i & 0 \\ \beta & f' \end{pmatrix}} C \oplus B' \xrightarrow{\begin{pmatrix} -q & 0 \\ \gamma & i' \end{pmatrix}} \Sigma A \oplus C' \xrightarrow{\begin{pmatrix} -\Sigma f & 0 \\ \Sigma\alpha & q' \end{pmatrix}} \Sigma B \oplus \Sigma A'$$

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A triangulated category $\mathcal{T}$ has a model if there is an exact equivalence $\mathcal{T} \simeq \operatorname{Ho}\mathcal{M}$ for some stable model category $\mathcal{M}$.

Example

The category of graded $\mathbb{F}_p[v_n, v_n^{-1}]$-modules, $|v_n| = 2p^n - 2$, has at least 2 non-equivalent models:

- *Differential graded $\mathbb{F}_p[v_n, v_n^{-1}]$-modules.*
- *$K(n)$-module spectra.*

Theorem (Schwede'05)

The stable homotopy category of spectra $\operatorname{Ho}\mathbf{Sp}$ admits a unique model up to Quillen equivalence.

Models for triangulated categories

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More generally we say that $\mathcal{T}$ has a model if there is an exact inclusion $\mathcal{T} \subset \operatorname{Ho} \mathcal{M}$.

Neeman defined a K-theory $K(\mathcal{T})$ for triangulated categories.

Theorem (Neeman'97)

Let $\mathcal{A}$ be an abelian category and let $\mathcal{T}$ be a triangulated category with a bounded t-structure with heart $\mathcal{A}$. If $\mathcal{T}$ admits a Waldhausen model then

$$K(\mathcal{A}) \simeq K(\mathcal{T}).$$

Example

$\mathcal{T} = \mathcal{D}^b(\mathcal{A}) \subset \mathcal{D}(\mathcal{A}) = \operatorname{Ho} \mathbf{Ch}(\mathcal{A})$.

Neeman's theorem can be used to obtain $K(\mathcal{A}) \simeq K(\mathcal{B})$ by embedding adequately two abelian categories $\mathcal{A}$, $\mathcal{B}$ in $\mathcal{T}$.

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A triangulated category without models

Theorem A

The category $\mathcal{F}(\mathbb{Z}/4)$ of finitely generated free $\mathbb{Z}/4$-modules has a unique triangulated structure with $\Sigma =$ identity and exact triangle

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4.$$

proof

Theorem B

There are not non-trivial exact functors

$$\mathcal{F}(\mathbb{Z}/4) \longrightarrow \operatorname{Ho}\mathcal{M},$$
$$\operatorname{Ho}\mathcal{M} \longrightarrow \mathcal{F}(\mathbb{Z}/4).$$

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Corollary

$\mathcal{F}(\mathbb{Z}/4)$ does not have models.

remarks

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Corollary

$\mathcal{F}(\mathbb{Z}/4)$ *does not have models.*

remarks

$\mathcal{F}(\mathbb{Z}/4)$ is triangulated

Two candidate triangle morphisms (α, β, γ) and $(\alpha', \beta', \gamma')$ are homotopic if there are morphisms (Θ, Φ, Ψ)

$$A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{q} \Sigma A$$

$$A' \xrightarrow[f']{} B' \xrightarrow[i']{} C' \xrightarrow[q']{} \Sigma A'$$

such that

$$\begin{aligned} \beta' - \beta &= \Phi i + f'\Theta, \\ \gamma' - \gamma &= \Psi q + i'\Phi, \\ \Sigma(\alpha' - \alpha) &= \Sigma(\Theta f) + q'\Psi. \end{aligned}$$

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$\mathcal{F}(\mathbb{Z}/4)$ is triangulated

Homotopic morphisms have isomorphic mapping cones.

A candidate triangle is contractible if the identity morphism is homotopic to the zero morphism.

The exact triangles in $\mathcal{F}(\mathbb{Z}/4)$ are the candidate triangles isomorphic to the direct sum of a contractible triangle and a triangle X_2 of the form

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Let us check that $\mathcal{F}(\mathbb{Z}/4)$ is triangulated.

- $A \to A \to 0 \to A$ is contractible.
- The translate of a contractible triangle is contractible. The translate of X_2 is X_2.
- Any morphism in $\mathcal{F}(\mathbb{Z}/4)$ is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} : W \oplus X \oplus Y \longrightarrow W \oplus X \oplus Z.$$

It can be extended to an exact triangle which is the direct sum of X_2 and the contractible triangle

$$W \oplus Y \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} W \oplus Z \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} Y \oplus Z \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} W \oplus Y.$$

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- $A \to A \to 0 \to A$ is contractible.
- The translate of a contractible triangle is contractible. The translate of X_2 is X_2.
- Any morphism in $\mathcal{F}(\mathbb{Z}/4)$ is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} : W \oplus X \oplus Y \longrightarrow W \oplus X \oplus Z.$$

It can be extended to an exact triangle which is the direct sum of X_2 and the contractible triangle

$$W \oplus Y \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} W \oplus Z \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} Y \oplus Z \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} W \oplus Y.$$

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$$\begin{tikzcd}
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\end{tikzcd}$$

for any $\delta\colon X \to Y$. Suppose that

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$$\underbrace{(\text{mapping cone of } 1_{L_2})}_{\text{contractible}} \oplus M_2 \oplus M_2 \oplus N_2 \oplus P_2, \quad \text{exact.}$$

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$\mathcal{F}(\mathbb{Z}/4)$ is triangulated

A candidate triangle in $\mathcal{F}(\mathbb{Z}/4)$

$$A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{q} A$$

is quasi-exact if

$$A \xrightarrow{f} B \xrightarrow{i} C \xrightarrow{q} A \xrightarrow{f} B$$

is an exact sequence of $\mathbb{Z}/4$-modules.

Example

X_2 is quasi-exact. Contractible triangles are quasi-exact.

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\end{tikzcd}
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contractible

quasi-exact

C is free. Let (Θ, Φ, Ψ) be a contracting homotopy for the upper row,

$$\gamma = \gamma' + (i'\beta - \gamma' i)\Phi.$$

Similarly if the first row is quasi-exact and the second row is contractible since $\mathbb{Z}/4$ is a Frobenius ring, so the duality functor $\mathrm{Hom}_{\mathbb{Z}/4}(-, \mathbb{Z}/4)$ preserves contractible triangles and quasi-exact triangles.

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Let T and T' be contractible triangles in $\mathcal{F}(\mathbb{Z}/4)$. Any commutative square between the first arrows of $X_2 \oplus T$ and $Y_2 \oplus T'$ can be extended to a morphism

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} : X_2 \oplus T \longrightarrow Y_2 \oplus T',$$

such that the mapping cone of $\varphi_{11} : X_2 \to Y_2$ is exact. Morphisms from or to contractible triangles are null-homotopic, so

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back remarks

$\mathcal{F}(\mathbb{Z}/4)$ is orthogonal to $\operatorname{Ho}\mathcal{M}$

We are going to define two kinds of objects in a triangulated category $\mathcal{T}$ according to the cofiber of $2 \cdot 1_X \colon X \to X$.

Example

If S is the sphere spectrum there is an exact triangle in $\operatorname{Ho}\mathbf{Sp}$

$$S \xrightarrow{2\cdot 1_S} S \xrightarrow{i} S/2 \xrightarrow{q} \Sigma S,$$

where $S/2$ is the mod 2 Moore spectrum. The map

$$2 \cdot 1_{S/2} \colon S/2 \to S/2$$

is the composite

$$S/2 \xrightarrow{q} \Sigma S \xrightarrow{\eta} S \xrightarrow{i} S/2,$$

where η is the stable Hopf map, which satisfies $2 \cdot \eta = 0$.

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$\mathcal{F}(\mathbb{Z}/4)$ is orthogonal to Ho $\mathcal{M}$

Definition

Let $A \in \mathcal{T}$ and let

$$A \xrightarrow{2\cdot 1_A} A \xrightarrow{i} C \xrightarrow{q} \Sigma A$$

be an exact triangle. A Hopf map for A is a map $\eta\colon \Sigma A \to A$ such that

$$\begin{aligned} 2 \cdot 1_C &= i\eta q, \\ 2 \cdot \eta &= 0. \end{aligned}$$

If A admits a Hopf map we say that A is hopfian.

Exact functors preserve Hopf maps and hopfian objects.

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$\mathcal{F}(\mathbb{Z}/4)$ is orthogonal to Ho $\mathcal{M}$

Proposition

If $\mathcal{T}$ admits a model then all objects are hopfian.

Proof.

Sp is "the free stable model category on one generator" S [Schwede-Shipley'02]. In particular for any object $A \in \operatorname{Ho}\mathcal{M}$ there is an exact functor

$$F_A\colon \operatorname{Ho}\mathbf{Sp} \longrightarrow \operatorname{Ho}\mathcal{M}$$

with $F_A(S) = A$, so A is hopfian as S. □

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Definition

An object $E \in \mathcal{T}$ is exotic if there is an exact triangle

$$E \xrightarrow{2\cdot 1_E} E \xrightarrow{2\cdot 1_E} E \xrightarrow{q} \Sigma E.$$

Example

$\mathbb{Z}/4$ is exotic in $\mathcal{F}(\mathbb{Z}/4)$. Indeed all objects in $\mathcal{F}(\mathbb{Z}/4)$ are exotic.

Exact functors preserve exotic objects.

$\mathcal{F}(\mathbb{Z}/4)$ is orthogonal to Ho $\mathcal{M}$

Definition

An object $E \in \mathcal{T}$ is exotic if there is an exact triangle

$$E \xrightarrow{2\cdot 1_E} E \xrightarrow{2\cdot 1_E} E \xrightarrow{q} \Sigma E.$$

Example

$\mathbb{Z}/4$ *is exotic in* $\mathcal{F}(\mathbb{Z}/4)$. *Indeed all objects in* $\mathcal{F}(\mathbb{Z}/4)$ *are exotic.*

Exact functors preserve exotic objects.

Proposition

If $X \in \mathcal{T}$ is both hopfian and exotic then $X = 0$.

Proof.

If $\eta\colon \Sigma X \to X$ is a Hopf map and

$$X \xrightarrow{2\cdot 1_X} X \xrightarrow{2\cdot 1_X} X \xrightarrow{q} \Sigma X$$

is exact then

$$2 \cdot 1_X = (2 \cdot 1_X)\eta q = 0,$$

therefore

$$X \xrightarrow{0} X \xrightarrow{0} X \xrightarrow{q} \Sigma X$$

is exact, so $X = 0$. □

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Proof of Theorem B.

All objects in Ho $\mathcal{M}$ are hopfian and all objects in $\mathcal{F}(\mathbb{Z}/4)$ are exotic. Therefore

$$F\colon \mathcal{F}(\mathbb{Z}/4) \longrightarrow \operatorname{Ho}\mathcal{M}$$

is an exact functor the image of F consists of objects which are both hopfian and exotic, so $F = 0$. Similarly for

$$F\colon \operatorname{Ho}\mathcal{M} \longrightarrow \mathcal{F}(\mathbb{Z}/4).$$

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Remarks

Theorems A and B are not only true for $R = \mathbb{Z}/4$ but for any commutative local ring R with maximal ideal $\mathfrak{m} = (2) \neq 0$ such that $\mathfrak{m}^2 = 0$. For instance $R = W_2(k)$, k a perfect field of char $k = 2$.

Let k be a field of char $k = 2$. The category $\mathcal{F}(k[\varepsilon]/\varepsilon^2)$ of finitely generated free modules over the ring of dual numbers $k[\varepsilon]/\varepsilon^2$ has a unique triangulated structure with $\Sigma =$ identity and exact triangle

$$k[\varepsilon]/\varepsilon^2 \xrightarrow{\varepsilon} k[\varepsilon]/\varepsilon^2 \xrightarrow{\varepsilon} k[\varepsilon]/\varepsilon^2 \xrightarrow{\varepsilon} k[\varepsilon]/\varepsilon^2.$$

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Proposition

The triangulated category $\mathcal{F}(k[\varepsilon]/\varepsilon^2)$ is exact equivalent to $\mathcal{D}^c(A)$, so it has a model given by differential graded right modules over a differential graded algebra A.

A is a DGA such that

$$H_0(A) = k[\varepsilon]/\varepsilon^2,$$

any right DG A-module M has $H_0(M)$ free as a $k[\varepsilon]/\varepsilon^2$-module, and the equivalence is given by

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Remarks

$A = k\langle a, u, v, v^{-1}\rangle / I$ with

$$|a| = |u| = 0, \quad |v| = -1.$$

The two-sided ideal I is generated by

$$a^2, \ au + ua + 1, \ av + va, \ uv + vu.$$

The differential is defined by

$$d(a) = u^2 v, \ d(u) = 0, \ d(v) = 0.$$

$$H_*(A) = k[x, x^{-1}] \otimes_k k[\varepsilon]/\varepsilon^2, \ \text{ with } \varepsilon = \{u\}, \ x = \{v\}.$$

We have a non-trivial Massey product

$$\langle \varepsilon, \varepsilon \cdot x, \varepsilon \rangle = 1 \quad \mod \varepsilon.$$

Given $y \in H_0(M)$ with $y \cdot \varepsilon = 0$ then

$$\langle y, \varepsilon, \varepsilon \cdot x \rangle \cdot \varepsilon = y \cdot \langle \varepsilon, \varepsilon \cdot x, \varepsilon \rangle = y \Rightarrow H_0(M) \text{ is free.}$$

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Theorem (Hovey-Lockridge'07)

Let R be a commutative ring. The category $\mathcal{F}(R)$ is triangulated with $\Sigma =$ identity if and only if R is a finite product of fields, rings of dual numbers over fields of characteristic 2, and local rings with $\mathfrak{m} = (2) \neq 0$ and $\mathfrak{m}^2 = 0$.

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There are many different kinds of models for triangulated categories:

- Stable model categories.
- Stable homotopy categories (Heller).
- Triangulated derivators (Grothendieck).
- Stable ∞-categories (Lurie).

In all these cases the free model in one generator is associated to the triangulated category of finite spectra, therefore Theorem B is also true for these kinds of models.

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The End

Thanks for your attention!