An application of A_∞-obstructions to representation theory

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8 May 2019

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Obstruction theory

Braces

We work over a perfect ground field k.

Let (X, d_X) be a cochain complex. Its endomorphism operad $\mathcal{E}_X$ is $\mathcal{E}_X(n) = \mathrm{Hom}(X \otimes \overset{n}{\cdots} \otimes X, X)$. It is a brace algebra with

$$x\{y_1, \dots, y_r\} = \sum \pm x(\dots, y_1, \dots, y_2, \dots\dots, y_r, \dots).$$

It is also a Lie algebra,

$$[x, y] = x\{y\} \pm y\{x\}.$$

Naive obstructions

An A_∞-algebra on X is a sequence of elements $m_n \in \mathcal{E}_X(n)$, $n \geq 2$, of degree $2 - n$ such that

$$d(m_n) = \sum_{p+q=n+1} m_p\{m_q\}.$$

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Given an A_{n-1}-algebra on X,

$$\sum_{p+q=n+1} m_p\{m_q\}$$

is an cocycle, which is a coboundary if and only if it can be extended to an A_n-algebra. Obstructions live in the cohomology of $\mathcal{E}_X$.

Homotopical interpretation

The operad $\mathcal{A}_\infty$ is free graded $\mathcal{F}(\mu_2, \mu_3, \dots)$ with differential

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The suboperads $\mathcal{A}_n \subset \mathcal{A}_\infty$ are free graded $\mathcal{F}(\mu_2, \dots, \mu_n)$ and each $\mathcal{A}_{n-1} \subset \mathcal{A}_n$ is a principal cofibration whose attaching map is defined by the previous summation,

$$\mathcal{F}(\Sigma^{-1}\mu_n) \xrightarrow{\text{attach}} \mathcal{A}_{n-1} \rightarrowtail \mathcal{A}_n$$

Homotopical interpretation

Take an A_{n-1}-algebra on X and consider the diagram

$$\begin{tikzcd} \mathcal{F}(\Sigma^{-1}\mu_n) \arrow[r, "\text{attach}"] \arrow[dr, "\simeq 0"'] & \mathcal{A}_{n-1} \arrow[r, tail] \arrow[d] & \mathcal{A}_n \arrow[dl, dashed] \\ & \mathcal{E}_X & \end{tikzcd}$$

The map $\downarrow$ classifies the A_{n-1}-algebra on X.

A null-homotopy for the map $\searrow$ amounts to a choice of m_n, which allows for the dashed extension $\swarrow$.

The minimal case $d_X = 0$

When $d_X = 0$ naive obstructions live in $\mathcal{E}_X$, but m_2 is associative since $m_2\{m_2\} = 0$, so $A = (X, m_2)$ is a graded algebra. Its Hochschild complex is $\mathcal{E}_X$ with differential $[m_2, -]$. Hochschild cohomology is (arity, degree) bigraded $\mathrm{HH}^{\star,*}(A, A)$.

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No m_n appears really in the n^{th} A_∞ equation, $n \geq 4$,

$$d(m_n) = 0 = \sum_{p+q=n+1} m_p\{m_q\} = [m_2, m_{n-1}] + \sum_{\substack{p+q=n+1\\ p,q\leq n-2}} m_p\{m_q\}.$$

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Hence m_n can be freely modified in a minimal A_n-algebra.

Given an A_{n-1}-algebra, $n \geq 4$, the last summation is a Hochschild cocycle, and it is a Hochschild coboundary if and only if it can be extended to an A_n-algebra after possibly replacing m_{n-1}.

Classical obstructions

These obstructions live in $\mathrm{HH}^{n,3-n}(A,A)$, $n \geq 4$. They go back to Kadeishvili'80 and Prouté'85.

For $n = 4$ the obstruction always vanishes.

For $n = 5$ it is

$$\frac{[\{m_3\},\{m_3\}]}{2} \in \mathrm{HH}^{5,-2}(A,A),$$

at least in $\operatorname{char} k \neq 2$, where the universal Massey product

$$\{m_3\} \in \mathrm{HH}^{3,-1}(A,A)$$

is well defined because $[m_2, m_3] = 0$ (Baues–Dreckmann'89, Benson–Krause–Schwede'04...).

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What is the homotopical interpretation of this obstruction theory?

Modules over operads

A module[1] over an operad $\mathcal{O}$ is a sequence $M = \{M(n)\}_{n\geq 0}$ equipped with composition products

$$M(p) \otimes \mathcal{O}(q) \xrightarrow{\circ_i} M(p+q-1) \xleftarrow{\circ_i} \mathcal{O}(p) \otimes M(q), \quad 1 \leq i \leq p,$$

satisfying the usual associativity equations. Any operad map $\mathcal{O} \to \mathcal{P}$ turns $\mathcal{P}$ into an $\mathcal{O}$-module.

[1]Defined by Markl'96, called linear modules by Baues–Jibladze–Tonks'97 and infinitesimal bimodules by Merkulov–Vallette'09.

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The category of $\mathcal{O}$-modules inherits an enriched abelian model category structure from cochain complexes. There is a Quillen pair

$$\mathcal{O}\text{-Mod} \underset{\text{forget}}{\overset{L_{\mathcal{O}}}{\rightleftarrows}} \mathcal{O} \downarrow \text{Operads}$$

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Homotopical interpretation

Recall that, in $\mathcal{A}_n$, $d(\mu_n) = [\mu_2, \mu_{n-1}] + \sum_{\substack{p+q=n+1 \\ p,q \leq n-2}} \mu_p\{\mu_q\}$.

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Let $\mathcal{B}_{n-2,2}$ be the free graded $\mathcal{A}_2$-module $\mathcal{F}_{\mathcal{A}_2}(\mu_{n-1}, \mu_n)$ with

$$d(\mu_n) = [\mu_2, \mu_{n-1}].$$

For $n \geq 4$, there is a cofiber sequence in $\mathcal{A}_2 \downarrow \mathsf{Operads}$

$$L_{\mathcal{A}_2}\Sigma^{-1}\mathcal{B}_{n-2,2} \xrightarrow{\text{attach}} \mathcal{A}_{n-2} \rightarrowtail \mathcal{A}_n$$

where the attaching map is given by

$$\begin{aligned} \Sigma^{-1}\mu_{n-1} &\mapsto \sum_{p+q=n} \mu_p\{\mu_q\}, \\ \Sigma^{-1}\mu_n &\mapsto \sum_{\substack{p+q=n+1 \\ p,q \leq n-2}} \mu_p\{\mu_q\}. \end{aligned}$$

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The map $\downarrow$ classifies the underlying A_{n-2}-algebra.

A null-homotopy for the map $\searrow$ amounts to a modification of m_{n-1} with a compatible choice of m_n, which allows for the dashed extension $\swarrow$.

New obstructions

For $n \geq 2s$, there are similar cofiber sequences in $\mathcal{A}_s \downarrow \mathsf{Operads}$

$$L_{\mathcal{A}_s}\Sigma^{-1}\mathcal{B}_{n-s,s} \xrightarrow{\text{attach}} \mathcal{A}_{n-s} \rightarrowtail \mathcal{A}_n$$

based on the fact that, in $\mathcal{A}_n$,

$$d(\mu_n) = \sum_{\substack{p+q=n+1 \\ p\leq s}} [\mu_p, \mu_q] + \sum_{\substack{p+q=n+1 \\ p,q\leq n-s}} \mu_p\{\mu_q\}.$$

They can be used to define obstructions to extending an A_{n-1}-algebra to an A_n-algebra after possibly replacing $m_{n-s+1}, \ldots, m_{n-1}$.

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They can be used to define obstructions to extending an A_{n-1}-algebra to an A_n-algebra after possibly replacing $m_{n-s+1}, \ldots, m_{n-1}$. **Where do these obstructions live?**

A spectral sequence

The tower of operads

$$\cdots \rightarrowtail \mathcal{A}_{n-1} \rightarrowtail \mathcal{A}_n \rightarrowtail \cdots$$

gives rise to a tower of fibrations

$$\cdots \twoheadleftarrow \mathrm{Map}(\mathcal{A}_{n-1}, \mathcal{E}_X) \twoheadleftarrow \mathrm{Map}(\mathcal{A}_n, \mathcal{E}_X) \twoheadleftarrow \cdots$$

whose Bousfield–Kan spectral sequence is given by the Hochschild complex in page 1,

$$E_1^{p,q} = \mathcal{E}_X(p+2)^{-q}, \qquad d_1 = [m_2, -],$$

and by Hochschild cohomology in page 2,

$$E_2^{p,q} = \mathrm{HH}^{p+2,-q}(A, A), \qquad p > 0.$$

The term $E_s^{p,q}$ contributes to $\pi_{q-p}\,\mathrm{Map}(\mathcal{A}_\infty, \mathcal{E}_X)$.

Obstructions recipients

The classical obstructions live in $\mathrm{HH}^{n,3-n}(A,A) = E_2^{n-2,n-3}$, $n \geq 4$, and the new ones live in $E_s^{n-2,n-3}$, $n \geq 2s$.[2]

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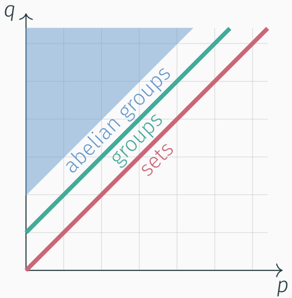
Caveat! The spectral sequence is only defined for $0 \leq p \leq q$,



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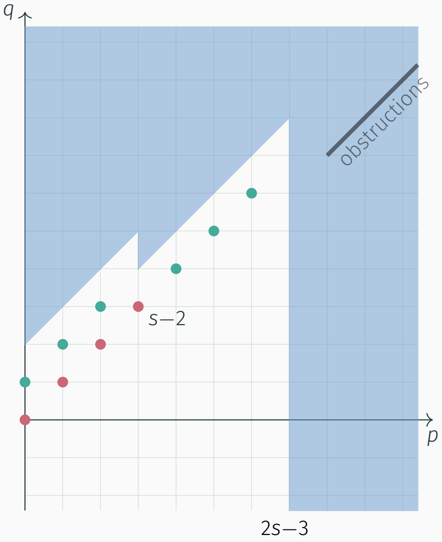
Obstructions recipients

We have extended it in each page E_s, following Bousfield'89,



Sets

Abelian groups

Vector spaces

Obstructions recipients

Caveat! The spectral sequence would require a base point in $\mathsf{Map}(\mathcal{A}_\infty, \mathcal{E}_X)$, i.e. an A_∞-algebra on X. If we only have an A_n-algebra, we can still define it up to page $\lfloor \frac{n+1}{2} \rfloor$.

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Theorem

The second differential of the spectral sequence is $d_2 = [\{m_3\}, -]$.

Here we use the Gerstenhaber bracket in Hochschild cohomology. In $\operatorname{char} k = 2$ there is an exceptional d_2 which must be dealt with separately.

Triangulated categories

Definition

A triangulated category is an additive category $\mathscr{T}$ equipped with a self-equivalence

$$\Sigma\colon \mathscr{T} \xrightarrow{\sim} \mathscr{T},$$

called suspension, and with diagrams

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{q} \Sigma X,$$

called exact triangles, satisfying the usual properties of cofiber sequences in the stable homotopy category or in a derived category.

Any DG-category $\mathscr{A}$ gives rise to a triangulated category $D^c(\mathscr{A})$, the derived category of compact $\mathscr{A}$-modules.

A Morita equivalence, in the sense of Tabuada, is a DG-functor $\mathscr{A} \to \mathscr{B}$ which induces an equivalence $D^c(\mathscr{A}) \simeq D^c(\mathscr{B})$.

We consider the set

$$\mathrm{ETC}(\mathscr{T}, \Sigma)$$

of Morita equivalence classes of DG-categories such that $D^c(\mathscr{A}) \simeq \mathscr{T}$ as suspended categories[3].

[3] ETC stands for *enhanced triangulated categories*.

Finiteness

An additive category $\mathscr{T}$ is finite if:

- idempotents split in $\mathscr{T}$,
- $\dim \mathscr{T}(X, Y) < \infty$ for any pair of objects,
- there are finitely many indecomposables $X_1, \ldots, X_n$ up to isomorphism.

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Lemma

There is an equivalence $\mathscr{T} \simeq \operatorname{proj}(\Lambda)$ where $\Lambda = \operatorname{End}_{\mathscr{T}}(X_1 \oplus \cdots \oplus X_n)$ and any self-equivalence is the restriction of scalars along an algebra automorphism $\sigma \colon \Lambda \cong \Lambda$.

Frobenius algebras

If $\mathscr{T}$ admits a triangulated structure then Λ is a Frobenius algebra by Freyd'66, in particular projective and injective Λ-modules coincide.

Two Λ-module maps $f, g \colon M \to N$ are homotopic if $f - g$ factors through an injective-projective object.

The homotopy category of Λ-modules is called the stable category. It is triangulated with $\Sigma = \Omega^{-1}$. The syzygy $\Omega(M)$ of a Λ-module M is the kernel of a projective cover

$$\Omega(M) \hookrightarrow P \twoheadrightarrow M.$$

Main theorem

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The enveloping algebra $\Lambda^{\mathrm{op}} \otimes \Lambda$ is also Frobenius.

Theorem

$\mathscr{T}$ is an enhanced triangulated category if and only if $\Omega^3_{\Lambda^{\mathrm{op}}\otimes\Lambda}(\Lambda)$ is stably isomorphic to ${}_1\Lambda_\sigma$. In that case, any two enhancements are Morita equivalent.

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There were finite triangulated categories with no known enhancements, and others whose enhancements where not known to be unique (Amiot'07, Keller'18, Hanihara'18).

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Let us see how this follows from the new A_∞-obstruction theory.

Hochschild–Tate cohomology

Since $\Lambda^{\mathrm{op}} \otimes \Lambda$ is also Frobenius, Λ admits a complete injective-projective resolution P_* as a Λ-bimodule

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[rr] \arrow[dr, two heads] & & P_{-1} \arrow[r] & P_{-2} \arrow[r] & \cdots \\
& & & & \Lambda \arrow[ur, hook] & & &
\end{tikzcd}$$

The cohomology of P_* with coefficients in a Λ-bimodule M is the Hochschild–Tate cohomology (Eu–Schedler'09)

$$\widehat{\mathrm{HH}}^n(\Lambda, M)$$

which coincides with Hochschild cohomology for $n > 0$.

Edge units

We consider the graded algebra

$$\Lambda(\sigma) = \frac{\Lambda\langle t^{\pm 1}\rangle}{(t\lambda - \sigma(\lambda)t)}, \qquad |t| = -1,$$

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An element in

$$\mathrm{HH}^{\star,*}(\Lambda(\sigma), \Lambda(\sigma))$$

is an edge unit if it maps to a unit in

$$\widehat{\mathrm{HH}}^{\star,*}(\Lambda, \Lambda(\sigma)).$$

The latter may have units in arbitrary bidegree, while the former only in $\star = 0$.

Enhanced triangulated structures

An enhanced triangulated structure on $(\mathscr{T}, \Sigma)$ is an A_∞-algebra $(\Lambda(\sigma), m_2, m_3, \dots)$ such that:

- m_2 is the product of $\Lambda(\sigma)$,
- $\{m_3\} \in \mathrm{HH}^{3,-1}(\Lambda(\sigma), \Lambda(\sigma))$ is an edge unit.

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A gauge equivalence is an A_∞-quasi-isomorphism with identity linear part.

We denote by $\mathrm{ETS}(\mathscr{T}, \Sigma)$ the set of gauge equivalence classes of enhanced triangulated structures, which carries a right action of $\mathrm{Aut}(\Lambda(\sigma))$ given by

$$m_n^g = g^{-1} m_n(g, \dots, g).$$

Relating enhancements

We have considered two kinds of enhanced triangulations on $\mathscr{T}$, given by DG-categories $\mathscr{A}$ with $D^c(\mathscr{A}) \simeq \mathscr{T}$ and A_∞-algebras on $\Lambda(\sigma)$, respectively.

Theorem

There is a bijection

$$\mathrm{ETS}(\mathscr{T}, \Sigma)/\operatorname{Aut}(\Lambda(\sigma)) \cong \mathrm{ETC}(\mathscr{T}, \Sigma)$$

sending each $(\Lambda(\sigma), m_2, m_3, \dots)$ to any A_∞-isomorphic DG-algebra.

Applying the new obstruction theory

Theorem

There is a bijection between $\mathsf{ETS}(\mathscr{T}, \Sigma)$ and the set of edge units $u \in \mathsf{HH}^{3,-1}(\Lambda(\sigma), \Lambda(\sigma))$ satisfying

$$\frac{[u, u]}{2} = 0.$$

It maps $(\Lambda(\sigma), m_2, m_3, \dots)$ to $\{m_3\}$.

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Idea of the proof.

Let m_3 be a representative of u. By the equation, there is an A_5-algebra $(\Lambda_\sigma, m_2, m_3, m_4, m_5)$. We have to show that it extends to an A_∞-algebra in an essentially unique way, after possibly modifying m_4 and m_5.

Applying the new obstruction theory

Cont.

We are going to see that $E_3^{p,q}$ vanishes for $p \geq 2$, hence all obstructions living therein vanish.

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On the one hand, multiplication by u in $\mathrm{HH}^{p,q}(\Lambda(\sigma), \Lambda(\sigma))$ is an isomorphism for $p \geq 2$ since it is an edge unit.

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On the one hand, multiplication by u in $\mathrm{HH}^{p,q}(\Lambda(\sigma), \Lambda(\sigma))$ is an isomorphism for $p \geq 2$ since it is an edge unit.

On the other hand, the Euler class $\delta \in \mathrm{HH}^{1,0}(\Lambda(\sigma), \Lambda(\sigma))$ satisfies

$$u \cdot x = [u, \delta \cdot x] + \delta \cdot [u, x]$$

for any x. Hence multiplication by u in Hochschild cohomology is null-homotopic for the differential $[u, -]$. This suffices by the computation of d_2. $\square$

Final steps

Proposition

The natural map

$$\mathrm{HH}^{3,-1}(\Lambda(\sigma), \Lambda(\sigma)) \longrightarrow \widehat{\mathrm{HH}}^{3,-1}(\Lambda, \Lambda(\sigma))$$

induces a bijection from set of edge units u in the source satisfying $\frac{[u,u]}{2} = 0$ to the set of units in the target.

Proposition

The set $\widehat{\mathrm{HH}}^{3,-1}(\Lambda, \Lambda(\sigma))^{\times} / \mathrm{Aut}(\Lambda(\sigma))$ is a singleton.

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