MODULI SPACES OF DG-ALGEBRAS

Algebra and Geometry Meeting
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Universidad de Sevilla

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$X_F = \operatorname{Spec} R$ moduli scheme of associative algebra structures on F

$$R = \frac{\Bbbk[c_{ij}^{k}]}{\left(\sum_{m=1}^{n} c_{ij}^{m} c_{mk}^{l} - c_{im}^{l} c_{jk}^{m}\right)}, \qquad X \subset \mathbb{A}^{n^3}.$$

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$Y_F = \operatorname{Spec} S$ moduli scheme of unital associative algebra structures on F

$$S = \frac{R[a_i]}{\left(\sum_{j=1}^{n} a_j c_{ji}^k - \delta_{ik}, \quad \sum_{j=1}^{n} c_{ij}^k a_j - \delta_{ik}\right)}, \qquad Y \subset \mathbb{A}^{n^3+n}.$$

Classical moduli schemes of algebras

There is a canonical map $Y_F \to X_F$ given by forgetting the unit. It is induced by the inclusion $R \subset S$.

Proposition (Gabriel'74)

The map $Y_F \to X_F$ is a Zariski open immersion.

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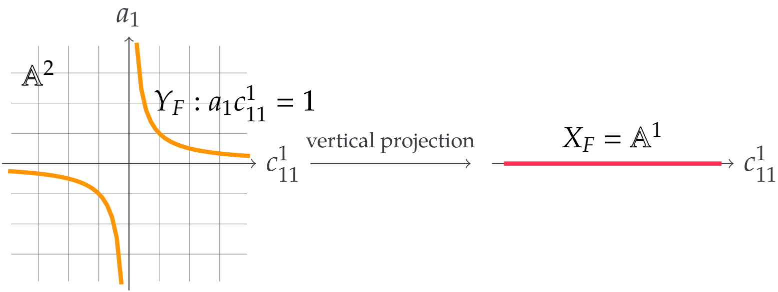
Proposition (Gabriel'74)

The map $Y_F \to X_F$ is a Zariski open immersion.

The morphism $R \to S$ is:

- finitely presented,
- flat,
- an epimorphism $S \amalg_R S = S \otimes_R S \cong S$.

Example $n = 1$



a_1
$\mathbb{A}^2$
$Y_F : a_1 c^1_{11} = 1$
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vertical projection
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Functors of points

$\mathrm{Aff}_{\mathbb{k}}^{\mathrm{op}}$ category of commutative $\mathbb{k}$-algebras.

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We are rather interested in them up to isomorphism.

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Corollary

The induced map $Y_F/\,\mathrm{GL}_F \to X_F/\,\mathrm{GL}_F$ is an affine Zariski open immersion.

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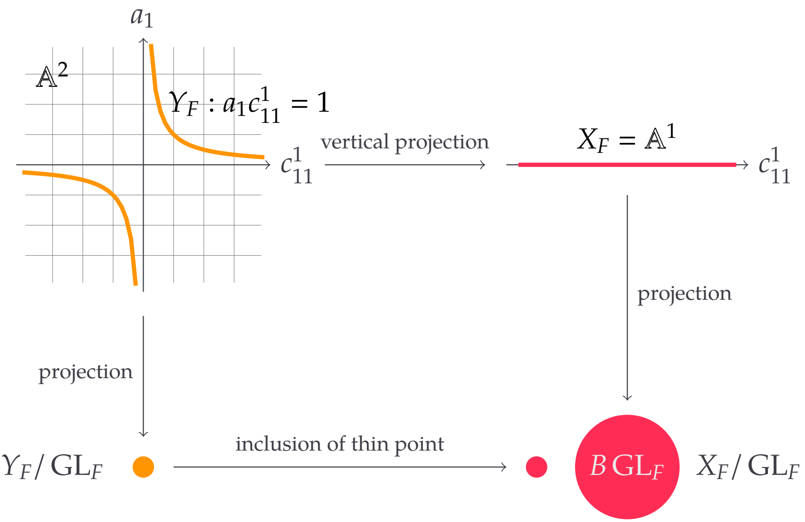
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Consider the classifying stack of automorphisms of F,

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We have a cartesian diagram,

$$\begin{tikzcd}
X_P \arrow[rr, "\text{structure map}"] \arrow[d, "\text{inclusion of discrete subcategory}"'] \arrow[drr, phantom, "\text{pullback}"] & & \operatorname{Spec} A \arrow[d, "\text{classified by a rank } n \text{ projective } A\text{-module } P"] \\
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Similarly with Y, so the fibers of $Y_F/\,\mathrm{GL}_F \to X_F/\,\mathrm{GL}_F$ coincide with the fibers of the maps $Y_P \to X_P$.

Complicial algebraic geometry [Toën–Vezzosi'08]

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A presheaf $F\colon \mathrm{Aff}_{\mathbb{k}}^{\mathrm{op}} \to \mathrm{Spaces}$ is a stack if:

- F takes quasi-isomorphisms to weak equivalences.
- F preserves products.
- Descent condition w.r.t. the strongly étale topology.

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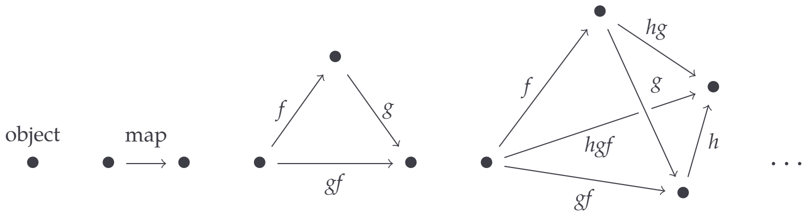
Affine stacks $\operatorname{Spec} A = \operatorname{Map}_{\mathrm{Aff}_{\mathbb{k}}^{\mathrm{op}}}(A, -)$ are stacks.

Spaces vs categories

Sometimes our stacks take values in categories and we compose with the geometric realization functor

$$|\cdot|\colon \text{Categories} \longrightarrow \text{Spaces},$$

which is essentially defined by taking chains of composable maps to simplices.



Complicial moduli stacks of DG-algebras

We consider the moduli stacks of (unital) associative DG-algebras on a perfect complex P [Toën–Vezzosi'08].

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X_P/\mathrm{GL}_P\colon\ \mathrm{Aff}_{\Bbbk}^{\mathrm{op}} &\longrightarrow \mathrm{Spaces},\\
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Flatness is essentially automatic.

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What are the fibers of the following map?

$$X_P/\,\mathrm{GL}_P \xrightarrow{\text{forgets the algebra structure}} B\,\mathrm{GL}_P$$

A_∞-algebras

Definition (Stasheff'63)

An A_∞-algebra is a complex C equipped with degree $n-2$ maps,

$$\mu_n \colon C \otimes \overset{n}{\cdots} \otimes C \longrightarrow C, \qquad n \geq 2,$$

satisfying

$$[d, \mu_n] = \sum_{\substack{p+q=n+1 \\ 1 \leq i \leq p}} \pm \mu_p \circ_i \mu_q.$$

A_∞-algebras

The first equations look as follows,

$$d\mu_2(x, y) = \mu_2(d(x), y) + (-1)^{|x|}\mu_2(x, d(y))$$

Leibniz rule,

$$[d, \mu_3](x, y, z) = \mu_2(\mu_2(x, y), z) - \mu_2(x, \mu_2(y, z))$$

homotopy associativity,

$$\dots$$

In particular $H_*(C)$ is an associative algebra.

An A_∞-algebra structure on $\Sigma^m\mathbb{k}$ with degree m generator $\mathbf{e}$ is given by

$$\mu_n(\mathbf{e},\dots,\mathbf{e}) = c_n\mathbf{e} \quad \text{structure constants} \quad |c_n| = n - 2 + mn - m$$

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$X_{\Sigma^m \Bbbk} = \operatorname{Spec} R$ moduli stack of A_∞-algebra structures on $\Sigma^m \Bbbk$

$$R = (\Bbbk[c_n], d).$$

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Replacing $\Sigma^m\Bbbk$ with a perfect complex P we still obtain an affine stack X_P.

A_∞-algebra structures

Theorem (Rezk'96, Toën–Vezzosi'08, M'14)

We have a cartesian diagram,

$$\begin{tikzcd} X_Q \arrow[rr, "\textit{structure map}"] \arrow[d, "\textit{inclusion of discrete subcategory}"'] & \arrow[d, phantom, "\textit{pullback}"] & \operatorname{Spec} A \arrow[d, "\textit{classified by a perfect A-module } Q"] \\ X_P / \mathrm{GL}_P \arrow[rr, "\textit{forgets the algebra structure}"'] & & B\,\mathrm{GL}_P \end{tikzcd}$$

Operads

Definition (Stasheff'63, May'72, Boardman–Vogt'73)

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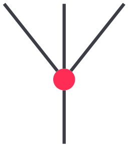

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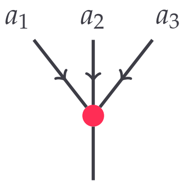

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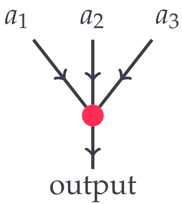

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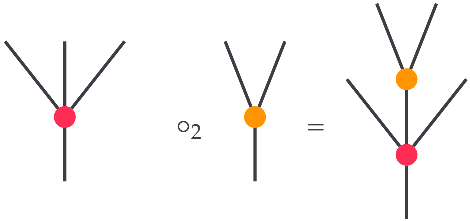

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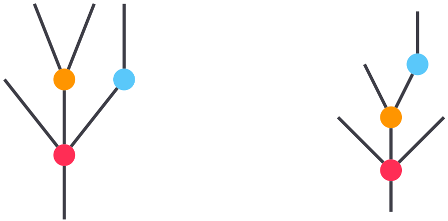
- \+ *associativity and unitality laws.*



Example

The endomorphism operad *of a complex C is given by*

$$\mathrm{End}_{\Bbbk}(C) = \{\mathrm{Hom}_{\Bbbk}(C \otimes \overset{n}{\cdots} \otimes C, C)\}_{n \geq 0},$$

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Definition

A $\mathtt{P}$-algebra is a complex C equipped with a map $\mathtt{P} \to \mathrm{End}_{\Bbbk}(C)$*, or equivalently a sequence of maps*

$$\mathtt{P}(n) \otimes C\otimes \overset{n}{\cdots} \otimes C \longrightarrow C$$

satisfying some laws.

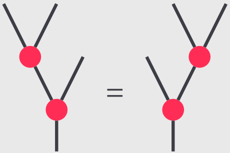
The operad of associative DG-algebras

Example (The associative operad)

The operad `As` *is generated by*

,

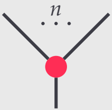
with relation

.

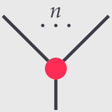
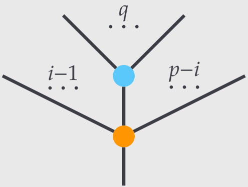
The operad of A_∞-algebras

Example (The A_∞-operad)

The operad $\mathtt{A}_\infty$ *is freely generated by*

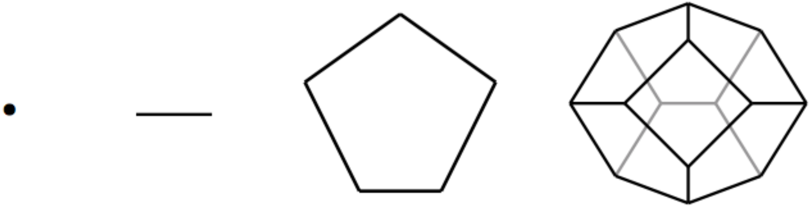
$\mu_n =$  , $\quad$ $degree = n - 2, \qquad n \geq 2,$

with differential

$d\Bigg($  $\Bigg) = \sum_{\substack{p+q=n+1 \\ 1 \leq i \leq p}} \pm$  .

A_∞-algebras and associahedra

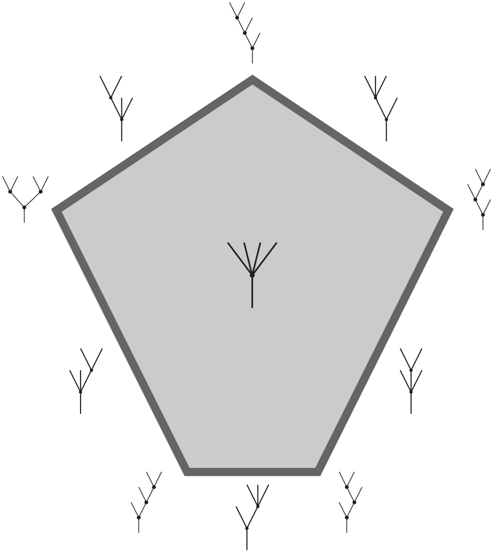
Stasheff showed that $\mathtt{A}_\infty(n)$ is the complex of cellular chains on the n^{th} associahedron K_n, $n \geq 2$,



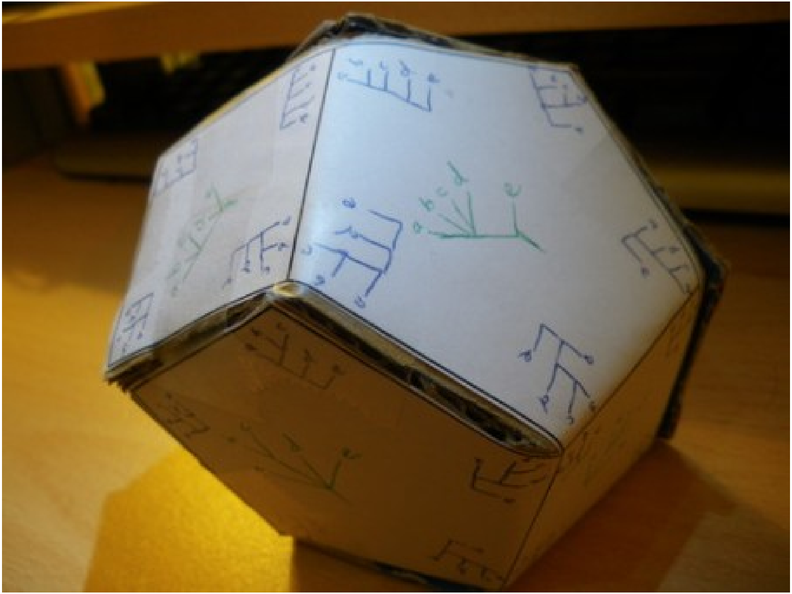
The element $\mu_n \in \mathtt{A}_\infty(n)_{n-2}$ is the top dimensional cell of the polytope K_n.

Associahedra

$K_4 =$



Associahedron K_5



Theorem (Hinich'97, Berger–Moerdijk'03, Lyubashenko'11, M'11…)

The category $\mathrm{Op}_{\Bbbk}$ *of operads carries a model structure with quasi-isomorphisms as weak equivalences and surjections as fibrations.*

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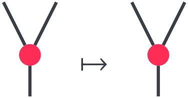
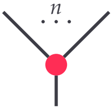


,



$\mapsto \quad 0, \quad n > 2.$

Operads and the moduli stack of A_∞-algebra structures

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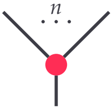


$\mapsto \quad 0, \quad n > 2.$

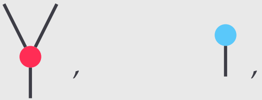
The stack X_P deserves its name since its functor of points is

$$\begin{aligned} X_P \colon \ \mathrm{Aff}_{\Bbbk}^{\mathrm{op}} &\longrightarrow \mathrm{Spaces}, \\ A &\mapsto \mathrm{Map}_{\mathrm{Op}_{\Bbbk}}(\mathtt{A}_\infty, \mathtt{End}_A(P \otimes A)). \end{aligned}$$

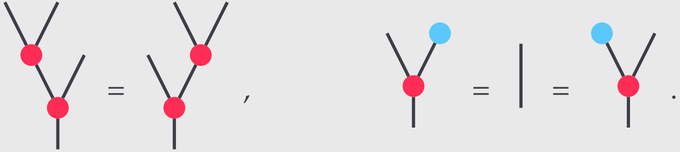
The operad of unital associative algebras

Example (The unital associative operad)

The operad `uAs` *is genetated by*



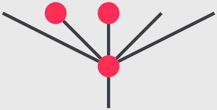
with relations



The unital A_∞ operad

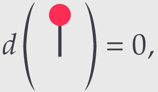
Definition (Fukaya–Oh–Ohta–Ono'09, Lyubashenko'11, M.–Tonks'14)

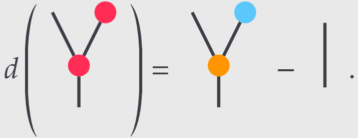
The operad $\mathtt{uA}_\infty$ *is the free extension of* $\mathtt{A}_\infty$ *generated by*



$$, \qquad \textit{degree} = \textit{leaves} + 2 \cdot \textit{corks} - 2,$$

with differential pictorially defined as in $\mathtt{A}_\infty$*, taking into accounts corks, with the following exceptions*





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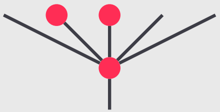 , $\quad$ *degree* $=$ *leaves* $+ 2 \cdot$ *corks* $- 2$,

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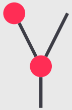
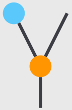
$d\Big($  $\Big) = 0,$ $\qquad$ $d\Big($  $\Big) =$  $-$.

Unital A_∞-algebras and unital associahedra

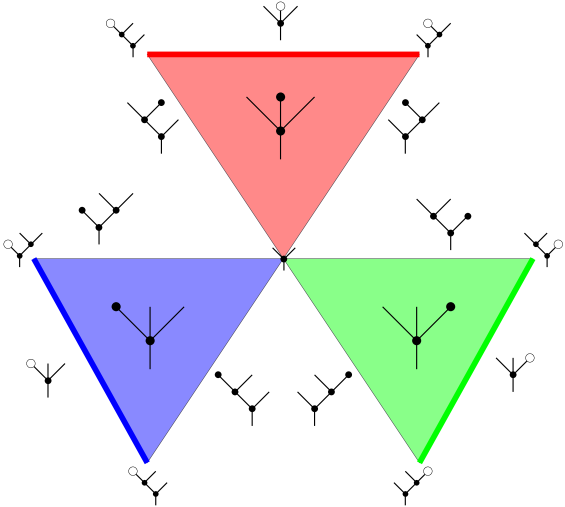
[M–Tonks'14] showed that $\mathtt{uA}_\infty(n)$ is the complex of cellular chains on the n^{th} unital associahedron K_n^u, $n \geq 0$, a contractible cell complex equipped with a cork filtration by finite subcomplexes $K_{n,m}^u$, $m \geq 0$,

Each new generator of $\mathtt{uA}_\infty$ corresponds to a cell of the form

$$K_{\text{leaves+corks}} \times [0,1]^{\text{corks}},$$

a product of an associahedron and a hypercube.

The piece of unital associahedron $K^u_{2,1}$



Theorem (Rezk'96, Toën–Vezzosi'08, M'14)

$$\begin{tikzcd}
Y_Q \arrow[rr, "\textit{structure map}"] \arrow[d, "\textit{inclusion of discrete subcategory}"'] \arrow[drr, phantom, "\textit{pullback}"] & & \operatorname{Spec} A \arrow[d, "\textit{classified by a perfect } A\textit{-module } Q"] \\
Y_P / \mathrm{GL}_P \arrow[rr, "\textit{forgets the algebra structure}"'] & & B\,\mathrm{GL}_P
\end{tikzcd}$$

We use the affine moduli stack of unital A_∞-algebra structures

$$\begin{aligned} Y_P \colon \mathrm{Aff}_{\Bbbk}^{\mathrm{op}} &\longrightarrow \mathrm{Spaces}, \\ A &\mapsto \mathrm{Map}_{\mathrm{Op}_{\Bbbk}}(\mathtt{uA}_\infty, \mathtt{End}_A(P \otimes A)). \end{aligned}$$

For $P = \Sigma^m \Bbbk$, $Y_{\Sigma^m \Bbbk} = \mathrm{Spec}(R[c_{n,S}], d)$, $n \geq 1$, $\varnothing \neq S \subset \{1, \ldots, n\}$.

Fibers of the forgetful map

The fibers of the map forgetting the unit

$$Y_P/\,\mathrm{GL}_P \longrightarrow X_P/\,\mathrm{GL}_P$$

are the fibers of the maps between affine stacks

$$Y_Q \longrightarrow X_Q$$

induced by the inclusion $\mathtt{A}_\infty \subset \mathtt{uA}_\infty$, so the former is affine.

The forgetful map is a homotopy monomorphism

Theorem (M'15)

The inclusion $\mathtt{A}_\infty \subset \mathtt{uA}_\infty$ is a homotopy epimorphism,

$$\mathtt{uA}_\infty \coprod_{\mathtt{A}_\infty} \mathtt{uA}_\infty \simeq \mathtt{uA}_\infty.$$

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Corollary

The map $Y_P/\mathrm{GL}_P \to X_P/\mathrm{GL}_P$ *is a homotopy monomorphism.*

This corollary also follows from [Lurie'14].

Finite presentation

We have $Y_P = \operatorname{Spec} S \to X_P = \operatorname{Spec} R$ induced by $R \subset S$ but S is not finitely generated over R.

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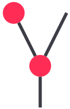
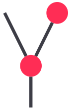
 ,

 .

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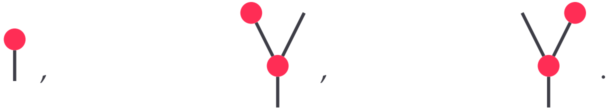

Theorem (Lyubashenko–Manzyuk'08, Lurie'14, Iwase)

Any $\mathtt{P}$-algebra extends to a unital A_∞-algebra.

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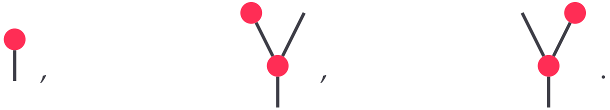

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Corollary

S is a homotopy retract of a subalgebra finitely generated over R, so $Y_P/\mathrm{GL}_P \to X_P/\mathrm{GL}_P$ is categorically finitely presented.

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- Simplicial $\Bbbk$-modules, derived algebraic geometry.
- Symmetric spectra, brave new algebraic geometry.
- Any context where the ground monoidal model category is simplicial, complicial, or spectral, satisfies the strong unit axiom and Schwede–Shipley's monoid axiom, is locally finitely presentable, and the tensor unit and the sources and targets of generating cofibrations are homotopically finitely presented.

MODULI SPACES OF DG-ALGEBRAS

Algebra and Geometry Meeting
Barcelona, 2–4 December 2015.

Fernando Muro

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