Uniqueness of models for triangulated categories in representation theory

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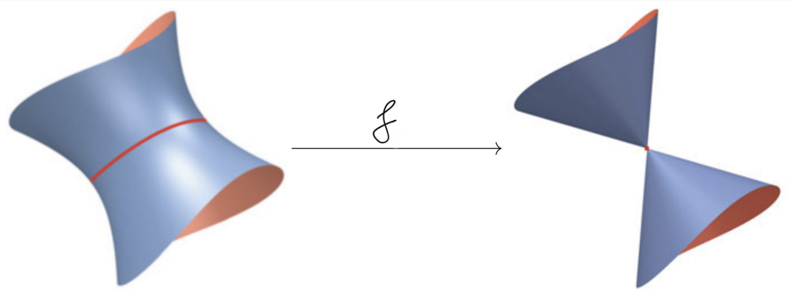
Contractible curves



Taken from Molina and Tosun 2020.

Contractible curves

Let $k = \mathbb{C}$ be the ground field and consider:

1. Y smooth quasi-projective 3-fold.
2. $i\colon C \hookrightarrow Y$ with $C^{\mathrm{red}} \cong \mathbb{P}^1$ a rational curve.
3. $f\colon Y \to X$ birational morphism satisfying:
   a) f restricts to $Y \setminus C \cong X \setminus p$ for $p \in X$.
   b) $f^{-1}(p) = C$.

Idea: study the singular point $p \in X$ by means of the resolution f.

The contraction f is a minimal model for X.

Deformations

Let us place ourselves in the following setting:

- A a k-algebra.
- M a right A-module.
- Λ a finite-dimensional local k-algebra.
- $\mathfrak{m} \subset \Lambda$ the maximal ideal, so $k \cong \Lambda/\mathfrak{m}$.

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An isomorphism of Λ-deformations is a Λ-A-bimodule isomorphism $\psi \colon N \xrightarrow{\cong} N'$ such that the following triangle commutes

$$\begin{array}{ccc} k \otimes_\Lambda N & \xrightarrow{k \otimes_\Lambda \psi} & k \otimes_\Lambda N' \\ & {\scriptstyle \varphi_N} \searrow \quad \swarrow {\scriptstyle \varphi_{N'}} & \\ & M & \end{array}$$

The contraction algebra

Let $\mathcal{A} = \mathsf{Mod}(A)$ be the category of right A-modules, $M \in \mathcal{A}$. The (non-commutative) deformation functor is defined as follows:

$$\begin{aligned} \mathsf{Def}_M^{\mathcal{A}} \colon \ \mathsf{Art} &\longrightarrow \mathsf{Set}, \\ \Lambda &\longmapsto \mathsf{Def}_M^{\mathcal{A}}(\Lambda) = \{\Lambda\text{-deformations of } M\} / \cong . \end{aligned}$$

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Take $\mathcal{A} = \mathsf{Qcoh}(Y)$ and $M = i_* \mathcal{O}_{\mathbb{P}^1}(-1)$ with $i \colon C \hookrightarrow Y$.

Theorem (Donovan and Wemyss 2016)

The functor $\mathsf{Def}_M^{\mathcal{A}} \cong \mathsf{Hom}_{\mathsf{Art}}(\underbrace{\Lambda_{\mathsf{con}}}_{\text{Contraction algebra}}, -)$ is representable.

This extends to $C^{\mathsf{red}} = \bigcup_{i=1}^n C_i$ with $C_i \cong \mathbb{P}^1$.

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Conjecture (Donovan and Wemyss 2016)

$$\widehat{\mathcal{O}}_{X,p} \cong \widehat{\mathcal{O}}_{X',p'} \iff \mathsf{D}^b(\Lambda_{\mathsf{con}}) \simeq \mathsf{D}^b(\Lambda'_{\mathsf{con}}).$$

August 2020 proved $\Rightarrow$.

The derived contraction algebra

The derived contraction algebra Γ pro-represents the corresponding derived deformation functor of Efimov, Lunts, and Orlov 2010:

- Γ is concentrated in ≤ 0.
- $H^0(\Gamma) = \Lambda_{\mathsf{con}} = \Lambda$.
- Γ is smooth, $\Gamma \in \mathsf{D}^c(\Gamma^{\mathsf{e}})$.
- Γ is 3-Calabi–Yau, $\mathsf{Hom}_{\Gamma^{\mathsf{e}}}(\Gamma, \Gamma^{\mathsf{e}}) = \Gamma[-3]$.

Let $\mathsf{D}^{fd}(\Gamma) \subset \mathsf{D}^c(\Gamma)$ be spanned by X with $\dim H^*(X) < \infty$.

The Amiot 2009 cluster DG category, which is the Drinfeld 2004 pre-triangulated quotient (Bondal and Kapranov 1991),

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Theorem (Hua and Keller 2021)

$$\widehat{\mathcal{O}}_{X,p} \cong \widehat{\mathcal{O}}_{X',p'} \iff \mathcal{C}_\Gamma^{dg} \simeq \mathcal{C}_{\Gamma'}^{dg}.$$

Can we recover $\mathcal{C}^{dg}_{\Gamma_{\mathsf{con}}}$ from Λ ?

The cluster category

The Amiot 2007 cluster category

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has the following properties:

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Properties 3+4+5 ⇒ $c \in \mathcal{T}$ is a $2\mathbb{Z}$-clsuter tilting object.

Recognition principle

A (small) triangulated category $\mathcal{T}$ is algebraic in the sense of Keller 2007 if $\mathcal{T} = \mathsf{D}^c(\mathcal{A})$ for some DG category $\mathcal{A}$. We then say that $\mathcal{A}$ is an enhancement or model of $\mathcal{T}$.

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Theorem (Jasso–M.'22)

1. The cluster category $\mathcal{C}_\Gamma$ admits a unique enhancement up to Morita equivalence.
2. Any small, Hom-finite, idempotent-complete, algebraic triangulated category $\mathcal{T}$ with a $2\mathbb{Z}$-cluster tilting object $c \in \mathcal{T}$ with endomorphism algebra $\mathcal{T}(c, c) = \Lambda$ and $\mathcal{T}(c, c[2]) \cong \Lambda$ as Λ-bimodules is

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Corollary

The Donovan and Wemyss 2016 conjecture holds.

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Then $\mathcal{A}$ is Morita equivalent to A,

$$H^*(A) = \mathcal{T}^*(c, c) = \bigoplus_{n \in \mathbb{Z}} \mathcal{T}(c, c[n]) = \bigoplus_{2|n} \Lambda = \Lambda[t^{\pm 1}], \qquad |t| = -2,$$

and A can be recovered from a minimal A_∞-model built on $\Lambda[t^{\pm 1}]$.

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Recovering the enhancement from the cohomology

A minimal A_∞-algebra structure on the graded algebra $\Lambda[t^{\pm 1}]$ is given by operations (Stasheff 1963):

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  $$\partial(m_6) + \frac{[m_4, m_4]}{2} = 0 \quad \text{so} \quad \frac{[\{m_4\},\{m_4\}]}{2} = 0.$$

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Theorem (Jasso–M.'22)

Given $x, y, z, t \in H^*(A) = \Lambda[t^{\pm 1}]$ whose Massey product exists,

$$-m_4(x, y, z, t) \in \langle x, y, z, t \rangle.$$

This Massey product coincides with the Toda bracket in $\mathcal{T}$ of

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The first property was believed to hold for any DG algebra, any minimal A_∞-model, Massey products of any length n, and m_n, but Buijs, Moreno-Fernández, and Murillo 2020 found counterexamples.

The restricted universal Massey product

The inclusion $j\colon \Lambda \hookrightarrow \Lambda[t^{\pm 1}]$ of the degree 0 part induces

$$j^*\colon \mathsf{HH}^{4,-2}(\Lambda[t^{\pm 1}], \Lambda[t^{\pm 1}]) \longrightarrow \mathsf{HH}^{4,-2}(\Lambda, \Lambda[t^{\pm 1}]),$$
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Since Λ is ungraded and $\Lambda[t^{\pm1}]$ is concentrated in even degrees,

$$\begin{aligned} \mathsf{HH}^{p,q}(\Lambda, \Lambda[t^{\pm1}]) &= \mathsf{HH}^{p}(\Lambda, \Lambda[t^{\pm1}]^{q}) \\ &= \begin{cases} \mathsf{HH}^{p}(\Lambda, \Lambda) = \mathsf{Ext}^{p}_{\Lambda^{\mathsf{e}}}(\Lambda, \Lambda), & q \text{ even}, \\ 0, & q \text{ odd}. \end{cases} \end{aligned}$$

The restricted universal Massey product

Since Λ is self-injective, it has complete projective-injective resolutions as a Λ-bimodule and we can define Tate Exts and Hochschild–Tate cohomology

$$\underline{\mathrm{HH}}^{\bullet,*}(\Lambda, \Lambda[t^{\pm 1}]) = \underline{\mathrm{Ext}}^{\bullet,*}_{\Lambda^e}(\Lambda, \Lambda[t^{\pm 1}]), \qquad \bullet, * \in \mathbb{Z}.$$

There is a natural comparison map

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Theorem (Jasso–M.'22)

The restricted universal Massey product $j^*\{m_4\} \in \mathsf{HH}^{4,-2}(\Lambda,\Lambda[t^{\pm 1}])$ is a unit in $\underline{\mathsf{HH}}^{\bullet,*}(\Lambda,\Lambda[t^{\pm 1}])$.

This follows from the connections between Toda brackets in $\mathcal{T}$, Massey products in $H^*(A)$, and the universal Massey product.

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Theorem (Kadeishvili 1988)

$\mathrm{HH}^{n+2,-n}(C) = 0$ for $n > 0 \Rightarrow C$ is intrinsically formal.

Intrinsic formality

A DG algebra B is formal if $B \simeq H^*(B)$.

A graded algebra C is intrinsically formal if, given a DG algebra B,

$$H^*(B) \cong C \implies B \simeq C.$$

Equivalently, given DG algebras B_1, B_2,

$$H^*(B_i) \cong C \implies B_1 \simeq B_2.$$

Theorem (Kadeishvili 1988)

$\mathrm{HH}^{n+2,-n}(C) = 0$ for $n > 0 \Rightarrow C$ is intrinsically formal.

If $H^*(B) \cong C$ is concentrated in even degrees and $\{m_4^B\} \neq 0 \in \mathrm{HH}^{4,-2}(C)$ then C is not intrinsically formal. In particular $\Lambda[t^{\pm 1}]$ is not intrinsically formal if Λ is not separable.

A separable example

If $\Lambda = k$ the algebraic triangulated category

$$\mathcal{T} = \mathsf{D}^c(k[t^{\pm 1}]) \simeq \mathsf{mod}(k) \times \mathsf{mod}(k)$$

has a basic $2\mathbb{Z}$-cluster tilting object

$$c = k[t^{\pm 1}] \mapsto (k, 0)$$

with (intrinsically formal graded) endomorphism algebra

$$\mathcal{T}(c, c) = \Lambda = k, \qquad \mathcal{T}^*(c, c) = \Lambda[t^{\pm 1}] = k[t^{\pm 1}],$$

by Kadeishvili 1988, since

$$\mathsf{HH}^{\bullet,*}(k[t^{\pm 1}]) = k[t^{\pm 1}, \delta]$$

with

$$\delta \in \mathsf{HH}^{1,0}(k[t^{\pm 1}])$$

the fractional Euler class, defined by $\delta_{|_\Lambda} = 0$ and $\delta(t) = -t$.

Massey formality

An even Massey algebra (C, m) is a graded algebra C concentrated in even degrees and

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Theorem (Jasso–M.'22)

$\mathrm{HH}^{n+2,-n}(C, m) = 0$ for $n > 2 \Rightarrow (C, m)$ is Massey formal.

Uniqueness

Theorem (Jasso–M.'22)

Up to isomorphism, there exists a unique even Massey algebra $(\Lambda[t^{\pm 1}], m)$ such that $j^*(m)$ is a Hochschild–Tate unit. Moreover, it is Massey formal. In particular $\mathcal{T}$ has a unique enhancement.

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Proof

1. $j^* m \in \mathsf{HH}^{4,-2}(\Lambda, \Lambda[t^{\pm 1}])$ is a unit in $\underline{\mathsf{HH}}^{\bullet,*}(\Lambda, \Lambda[t^{\pm 1}])$.

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3. The product with $j^* m$ in $\mathsf{HH}^{\bullet,*}(\Lambda, \Lambda[t^{\pm 1}])$ is an isomorphism for $\bullet > 0$ and surjective for $\bullet = 0$.

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3. The product with $j^* m$ in $\mathrm{HH}^{\bullet,*}(\Lambda, \Lambda[t^{\pm 1}])$ is an isomorphism for $\bullet > 0$ and surjective for $\bullet = 0$.
4. We have

$$\mathrm{HH}^{\bullet,*}(\Lambda[t^{\pm 1}]) \cong \mathrm{HH}^{\bullet,*}(\Lambda, \Lambda[t^{\pm 1}])[\delta].$$

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5. The product with m in $\mathrm{HH}^{\bullet,*}(\Lambda[t^{\pm1}])$ is an isomorphism for $\bullet > 1$ and surjective for $\bullet = 1$.
6. The product with m in $\mathrm{HH}^{\bullet,*}(\Lambda[t^{\pm1}])$ is nullhomotopic:

$$m \cdot x = [m, \delta \cdot x] + \delta \cdot [m, x].$$

To see it like a null-homotopy:

$$f(x) = \partial h(x) + h\partial(x),$$
$$f(x) = m \cdot x,$$
$$\partial(x) = [m, x],$$
$$h(x) = \delta \cdot x.$$

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   To see it like a null-homotopy:

$$\begin{aligned} f(x) &= \partial h(x) + h\partial(x), \\ f(x) &= m \cdot x, \\ \partial(x) &= [m, x], \\ h(x) &= \delta \cdot x. \end{aligned}$$

7. $\mathrm{HH}^{\bullet,*}(\Lambda[t^{\pm 1}]) = 0$ for $\bullet > 2$.

□

Triangulated Auslander–Iyama correspondence

Theorem (Jasso–M.'22)

Let k be a perfect field. There is a bijection between equivalence classes of pairs:

1. $(\mathcal{T}, c)$ where:
   a) $\mathcal{T}$ is a small, idempotent-complete, Hom-finite, algebraic triangulated category.
   b) c is a $2\mathbb{Z}$-cluster tilting object such that $\mathcal{T}(c, c[2]) \cong \mathcal{T}(c, c)$ as $\mathcal{T}(c, c)$-bimodules
2. Λ is a self-injective finite-dimensional algebra such that $\underline{\mathrm{HH}}^\bullet(\Lambda)$ has a unit in degree 4.

The bijection is given by $\Lambda = \mathcal{T}(c, c)$.

These triangulated categories admit a unique DG enhancement.

Surjectivity

A degree 4 unit in $\underline{\mathsf{HH}}^\bullet(\Lambda)$ is an element of

$$u \in \underline{\mathsf{HH}}^4(\Lambda) = \mathsf{Ext}^4_{\Lambda^e}(\Lambda, \Lambda) = \mathsf{HH}^{4,-2}(\Lambda, \Lambda[t^{\pm 1}]).$$

We can construct a DG algebra B such that

$$H^*(B) \cong \Lambda[t^{\pm 1}], \qquad j^*\{m_4\} \mapsto u \in \mathsf{HH}^{4,-2}(\Lambda, \Lambda[t^{\pm 1}]).$$

Then $\mathcal{T} = \mathsf{D}^c(B)$ has $2\mathbb{Z}$-cluster tilting object $c = B$ with the required properties.

Surjectivity

Theorem (Jasso–M.'22)

There exists a unique $m \in \mathsf{HH}^{4,-2}(\Lambda[t^{\pm1}], \Lambda[t^{\pm1}])$ such that

$$j^*(m) = u, \qquad \frac{[m, m]}{2} = 0.$$

Surjectivity

Theorem (Jasso–M.'22)

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Theorem (Jasso–M.'22)

Given an even Massey algebra (C, m), if $\mathsf{HH}^{n+1,-n}(C, m) = 0$ for $n > 7$ then there exists a DG algebra B with

$$H^*(B) \cong C, \qquad \{m_4\} \mapsto m.$$

Thanks for your attention! ☺

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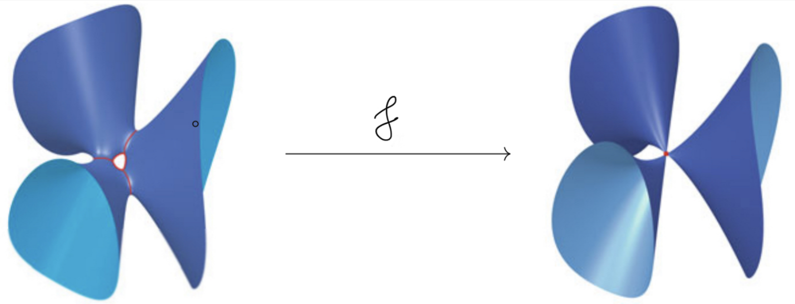
Contractible curves



Taken from Molina and Tosun 2020.

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Let k be a perfect field and $d \geq 1$. There is a bijection between equivalence classes of pairs:

1. $(\mathcal{T}, c)$ where:
   a) $\mathcal{T}$ a small algebraic triangulated category with finite-dimensional Hom's and split idempotents.
   b) c a basic $d\mathbb{Z}$-cluster tilting object.
2. $(\Lambda, [\sigma])$ where:
   a) Λ a basic finite-dimensional self-injective twisted $(d+2)$-periodic algebra.
   b) $[\sigma] \in \mathsf{Out}(\Lambda)$ such that $\Omega^{d+2}_{\Lambda^e}(\Lambda) \cong {}_1\Lambda_\sigma$ in $\underline{\mathsf{mod}}(\Lambda^e)$.

The bijection is given by $\Lambda = \mathcal{T}(c, c)$ and ${}_1\Lambda_\sigma = \mathcal{T}(c[d], c)$.[a]

These triangulated categories admit a unique DG enhancement.

[a] As objects, $c[d] = c$ but $[d]$ does not act like the identity on $\Lambda = \mathcal{T}(c, c)$.