Obstructions to Adams representability

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Adams' representability theorem

Let $\mathbf{T}$ be the stable homotopy category and $\mathbf{C}$ the full subcategory of compact spectra.

Definition

A cohomological functor $H\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Ab}$ is an additive functor which takes exact triangles to exact sequences.

Example

For any X in $\mathbf{T}$ the restricted representable functor $\mathbf{T}(-,X)_{|\mathbf{C}}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Ab}$ is cohomological.

Theorem (Adams' representability theorem, 1971)

[ARO] *Any cohomological functor $H\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Ab}$ is of the form $H \cong \mathbf{T}(-,X)_{|\mathbf{C}}$ for some X in $\mathbf{T}$.*

[ARM] *Any natural transformation $\mathbf{T}(-,X)_{|\mathbf{C}} \to \mathbf{T}(-,Y)_{|\mathbf{C}}$ is induced by a morphism $f\colon X \to Y$ in $\mathbf{T}$.*

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Adams' representability theorem

Theorem (Christensen'98)

Given two objects X and Y in $\mathbf{T}$, there is a short exact sequence where the kernel is the set of phantom maps,

$$\lim_{\substack{C\to X\\ C \text{ compact}}}{}^{1} \mathbf{T}(\Sigma C, Y) \rightarrowtail \mathbf{T}(X, Y) \twoheadrightarrow \lim_{\substack{C\to X\\ C \text{ compact}}} \mathbf{T}(C, Y)$$

Definition

A morphism $f\colon X \to Y$ in $\mathbf{T}$ is a phantom map if $\mathbf{T}(C, f) = 0$ for any C in $\mathbf{C}$.

Theorem (Neeman'97, Christensen–Strickland'98)

Phantom maps form a square zero ideal in $\mathbf{T}$.

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The stable homotopy category as an extension [CS'98]

Let $\mathrm{Mod}(\mathbf{C})$ be the abelian category of $\mathbf{C}$-modules, i.e. additive functors $\mathbf{C}^{\mathrm{op}} \to \mathbf{Ab}$.

Homological functors are the flat objects in $\mathrm{Mod}(\mathbf{C})$.

The restricted Yoneda functor

$$\begin{aligned} \mathbf{T} &\longrightarrow \mathrm{Mod}(\mathbf{C}), \\ X &\mapsto \mathbf{T}(-,X)_{|\mathbf{C}}, \end{aligned}$$

is, by Adams' representability theorem, full and essentially surjective onto the full subcategory $\mathrm{Flat}(\mathbf{C})$ of flat objects.

The subset of fantom maps in $\mathbf{T}(X,Y)$ is naturally isomorphic to $\mathrm{Ext}^1_{\mathbf{C}}(\mathbf{T}(-,X)_{|\mathbf{C}}, \mathbf{T}(-,Y)_{|\mathbf{C}})$, and there is a square-zero extension

$$\mathrm{Ext}^1_{\mathbf{C}} \rightarrowtail \mathbf{T} \twoheadrightarrow \mathrm{Flat}(\mathbf{C}),$$

which is classified by a Hochschild–Mitchell cohomology class

$$\{\mathrm{Ext}^1_{\mathbf{C}} \rightarrowtail \mathbf{T} \twoheadrightarrow \mathrm{Flat}(\mathbf{C})\} \in H^2(\mathrm{Flat}(\mathbf{C}), \mathrm{Ext}^1_{\mathbf{C}}).$$

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Can we generalize Adam's theorem to other categories?

If $\mathbf{T}$ is a compactly generated triangulated category we can take the subcategory $\mathbf{C}$ of compact objects, i.e. objects C in $\mathbf{T}$ such that

$$\mathbf{T}\left(C, \coprod_{i \in I} X_i\right) = \coprod_{i \in I} \mathbf{T}(C, X_i).$$

Theorem (Neeman'97)

If $\mathbf{C}$ is countable then Adams' representability theorem holds.

Example

- *Stable homotopy category.*
- *$D(R)$ if R is a countable ring, e.g. $\mathbb{Z}$.*
- *The stable motivic homotopy category over a finite-dimensional noetherian scheme with a cover by Spec(countable rings)* [Voevodsky'98, Naumann–Spitzweck'09].

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Theorem (Christensen–Keller–Neeman'01)

If k is a field of card $k \geq \aleph_2$ then $D(k\langle x, y\rangle)$ does not satisfy Adams' representability theorem, neither [ARM] nor [ARO].

Can we generalize Adam's theorem in another direction?

Let $\mathbf{T}$ be a well generated triangulated category and α a regular cardinal.

Recall that $\alpha = \aleph_0$ is a regular cardinal because any finite sum of finite cardinals is finite. In general, replace 'finite' with '$< \alpha$'.

Let $\mathbf{C}$ be the full subcategory of α-compact objects. An object C in $\mathbf{C}$ satisfies

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Failed theorem (Rosicky'05)

If $\mathbf{T}$ is a well generated triangulated category with models then for big enough regular cardinals α the following holds:

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If Rosický's theorem were true for $\mathbf{T}$, in addition to the obvious extensions of the previous results we would have:

Theorem (*Brown representability for the dual*, Neeman'09)

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Basic setting

$\mathbf{T} =$ a well generated triangulated category with translation functor Σ.

$\alpha =$ a regular cardinal.

$\mathbf{C} =$ a full essentially small subcategory of $\mathbf{T}$ closed under Σ, Σ^{-1}, and $\coprod$ of $< \alpha$ objects, and such that $\mathbf{C}$ generates $\mathbf{T}$.

$\mathrm{Mod}_\alpha(\mathbf{C}) =$ the graded abelian category of α-continuous $\mathbf{C}$-modules, i.e. functors $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Ab}$ such that, if $\operatorname{card} I < \alpha$, then

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grading

The restricted Yoneda functor

$$\begin{aligned} \mathbf{T} &\longrightarrow \mathrm{Mod}_\alpha(\mathbf{C}), \\ X &\mapsto \mathbf{T}(-, X)_{|\mathbf{C}}, \end{aligned}$$

which induces an equivalence between the completion of $\mathbf{C}$ in $\mathbf{T}$ by coproducts and direct summands and projective objects in $\mathrm{Mod}_\alpha(\mathbf{C})$.

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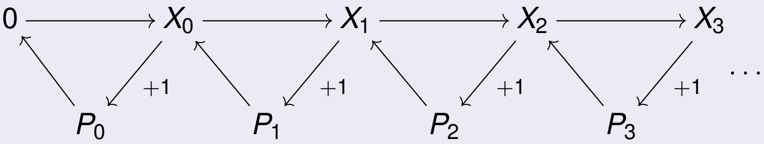
Postnikov resolutions

Definition (Benson–Krause–Schwede'04)

A Postnikov resolution of $F \colon \mathbf{C}^{op} \to \mathbf{Ab}$ *in* $\mathrm{Mod}_\alpha(\mathbf{C})$ *is a diagram of exact triangles in* $\mathbf{T}$



such that the induced complex

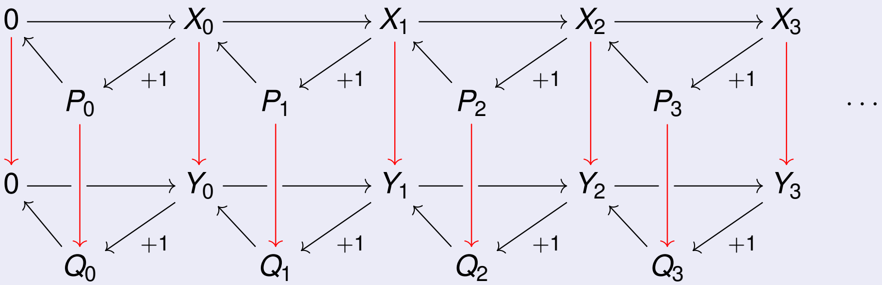
$$\cdots \leftarrow 0 \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow P_3 \leftarrow \cdots$$

is a projective resolution of F.

Postnikov resolutions

Definition

A *morphism* *of Postnikov resolutions is a commutative diagram in* **T**



Proposition

Given a Postnikov resolution of $F\colon \mathbf{C}^{op} \to \mathbf{Ab}$

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & & & X_0 & \longrightarrow & & & X_1 & \longrightarrow & & & X_2 & \longrightarrow & & & X_3 \\ & \nwarrow & & \swarrow_{+1} & & \nwarrow & & \swarrow_{+1} & & \nwarrow & & \swarrow_{+1} & & \nwarrow & & \swarrow_{+1} & & \cdots \\ & & P_0 & & & & P_1 & & & & P_2 & & & & P_3 & & \end{array}$$

we have a 'natural' isomorphism

$$F \cong \mathbf{T}(-, \operatorname{hocolim} X_n)_{|\mathbf{C}}.$$

In [BKS'04] when **C** consists of compact objects.

Postnikov resolutions

Proposition

- *Given X in $\mathbf{T}$, any projective resolution of $F = \mathbf{T}(-, X)_{|\mathbf{C}}$ can be completed to a Postnikov resolution, called good, such that* $\operatorname{hocolim} X_n = X$.
- *A morphism $f \colon X \to Y$ can be extended to a morphism between any two good Postnikov resolutions of $\mathbf{T}(-, X)_{|\mathbf{C}}$ and $\mathbf{T}(-, Y)_{|\mathbf{C}}$ inducing f on homotopy colimits.*

Corollary

- *A functor $F \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Ab}$ is of the form $H \cong \mathbf{T}(-, X)_{|\mathbf{C}}$ iff it admits a Postnikov resolution.*
- *A natural transformation $\mathbf{T}(-, X)_{|\mathbf{C}} \to \mathbf{T}(-, Y)_{|\mathbf{C}}$ is induced by a morphism $f \colon X \to Y$ iff it can be extended to a morphism between good Postnikov resolutions of $\mathbf{T}(-, X)_{|\mathbf{C}}$ and $\mathbf{T}(-, Y)_{|\mathbf{C}}$.*

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Postnikov resolutions

$$\big\downarrow$$

$\mathrm{Mod}_\alpha(\mathbf{C})$

$$
\begin{array}{c}
\text{Postnikov resolutions} \\
\downarrow \\
\vdots \\
\downarrow \\
(n+1)\text{-truncated Postnikov resolutions} \\
\downarrow \\
n\text{-truncated Postnikov resolutions} \\
\downarrow \\
\vdots \\
\downarrow \\
1\text{-truncated Postnikov resolutions} \\
= \\
\mathrm{Mod}_{\alpha}(\mathbf{C})
\end{array}
$$

The obstruction theory

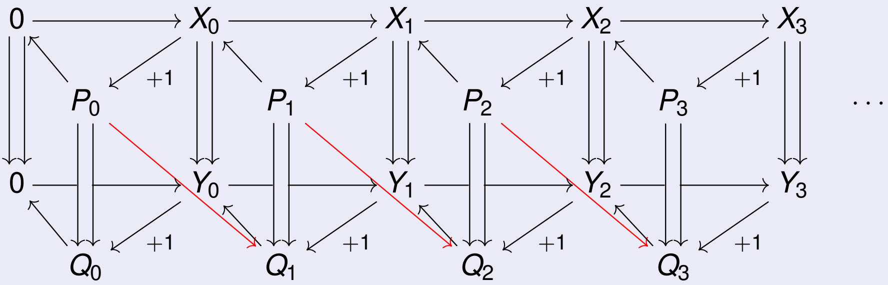
$$\begin{array}{c}\text{Postnikov resolutions}\\ \downarrow\\ \vdots\\ \downarrow\\ (n+1)\text{-truncated Postnikov resolutions}\\ \downarrow\\ n\text{-truncated Postnikov resolutions}\\ \downarrow\\ \vdots\\ \downarrow\\ 1\text{-truncated Postnikov resolutions}\\ =\\ \mathrm{Mod}_\alpha(\mathbf{C})\end{array}$$

Develop an obstruction theory to decide which objects and morphisms lift along these functors

Postnikov resolutions

Definition

A homotopy *between two morphisms of Postnikov resolutions is a sequence of morphisms* $P_n \to Q_{n+1}$, $n \geq 0$,



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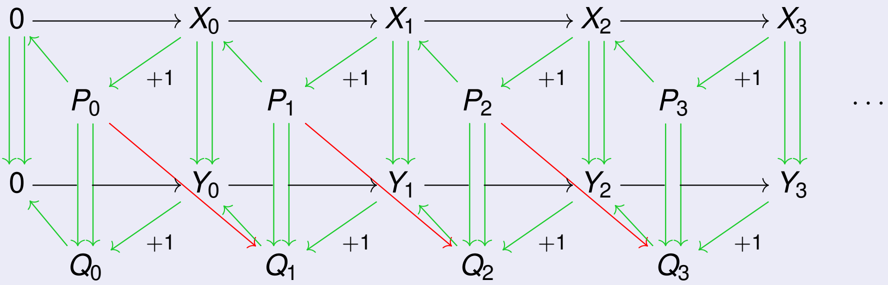

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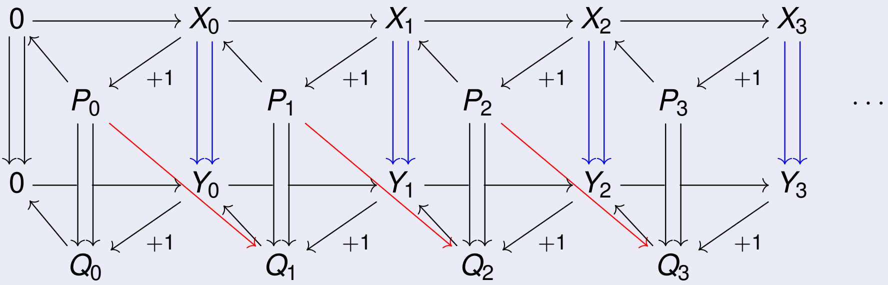

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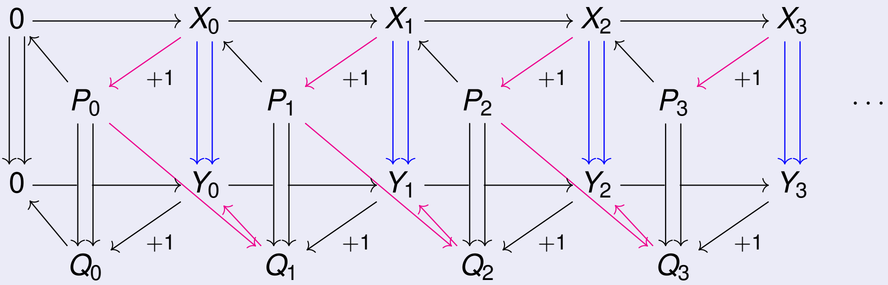

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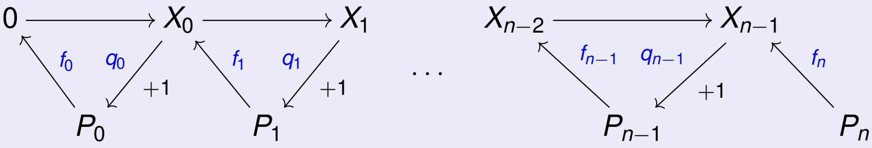
Truncated Postnikov resolutions

Definition

An n-truncated Postnikov resolution of $F\colon \mathbf{C}^{op} \to \mathbf{Ab}$ *in* $\mathrm{Mod}_\alpha(\mathbf{C})$ *consists of a diagram of n exact triangles in* $\mathbf{T}$ *with a tail*



and a projective resolution of F

$$P_0 \xleftarrow[q_0 f_1]{+1} P_1 \xleftarrow[q_1 f_2]{+1} \cdots\cdots \xleftarrow[q_{n-2} f_{n-1}]{+1} P_{n-1} \xleftarrow[q_{n-1} f_n]{+1} P_n \xleftarrow[d_{n+1}]{+1} P_{n+1} \xleftarrow[d_{n+2}]{+1} \cdots\cdots$$

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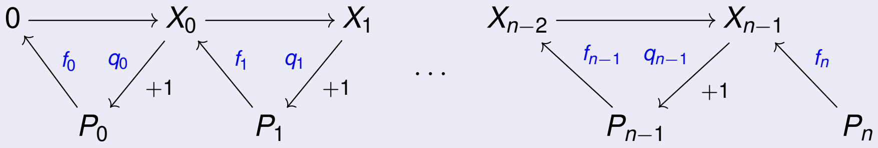

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Obstructions to lifting Postnikov resolutions

Theorem (Benson–Krause–Schwede'04)

Given an n-truncated Postnikov resolution of $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Ab}$*, there is an obstruction*

$$\kappa_{n+2} \in \mathrm{Ext}_{\mathbf{C}}^{n+2,-n}(F,F)$$

which vanishes iff it can be extended to an $(n+1)$*-truncated Postnikov resolution.*

Obstructions to lifting Postnikov resolutions

The first possibly non-trivial obstruction is for $n = 1$ and it only depends on F.

$$\kappa_3(F) \in \mathrm{Ext}^{3,-1}_{\mathbf{C}}(F, F).$$

definition

Proposition (*Naturality*, BKS'04)

Given a morphism $\tau\colon F \to G$ in $\mathrm{Mod}_\alpha(\mathbf{C})$,

$$\tau \circ \kappa_3(F) = \kappa_3(G) \circ \tau \in \mathrm{Ext}^{3,|\tau|-1}_{\mathbf{C}}(F, G).$$

This means that κ_3 is a class in Hochschild–Mitchell cohomology,

$$\kappa_3 \in H^{0,-1}(\mathrm{Mod}_\alpha(\mathbf{C}), \mathrm{Ext}^3_{\mathbf{C}}).$$

Corollary

If F has projective or injective dimension ≤ 2 then F is representable.

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Definition (Beligiannis'00)

The α*-pure global dimension of* $\mathbf{T}$ *is*

$$\alpha\text{-p. gl. dim}\,\mathbf{T} = \sup_{X \text{ in } \mathbf{T}} \text{p. d.}\,\mathbf{T}(-,X)_{|\mathbf{C}}.$$

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If α-p. gl. dim $\mathbf{T} \leq 2$ *then* [ARO] *holds.*

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Obstructions to lifting morphisms

Theorem

Given n-truncated Postnikov resolutions of $F, G\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Ab}$ and a morphism τ between its $(n-1)$-truncations, there is an obstruction

$$\theta_n(\tau) \in \mathrm{Ext}_{\mathbf{C}}^{n,1-n}(F, G)$$

which vanishes iff τ can be extended to a morphism between the given n-truncated Postnikov resolutions.

Moreover, there is an effective and transitive action of $\mathrm{Ext}_{\mathbf{C}}^{n,1-n}(F, F)$ on the set of isomorphism classes of n-truncated Postnikov resolutions of F with the same given $(n-1)$-truncation. The difference between two such n-truncated Postnikov resolutions is the obstruction to the realization of the identity in the common $(n-1)$-truncation.

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Obstructions to lifting morphisms

The first obstruction for $\tau\colon \mathbf{T}(-,X)_{|\mathbf{C}} \to \mathbf{T}(-,Y)_{|\mathbf{C}}$ is for $n = 2$ and only depends on X and Y,

$$\theta_2^{X,Y}(\tau) \in \mathrm{Ext}_{\mathbf{C}}^{2,-1}(\mathbf{T}(-,X)_{|\mathbf{C}}, \mathbf{T}(-,Y)_{|\mathbf{C}}).$$

definition

Proposition (*Derivation*)

Given composable morphism $\mathbf{T}(-,X)_{|\mathbf{C}} \xrightarrow{\tau} \mathbf{T}(-,Y)_{|\mathbf{C}} \xrightarrow{\sigma} \mathbf{T}(-,Z)_{|\mathbf{C}}$

$$\theta_2^{X,Z}(\sigma\circ\tau) = \theta_2^{Y,Z}(\sigma)\circ\tau + (-1)^{|\sigma|}\sigma\circ\theta_2^{X,Y}(\tau).$$

This means that, if $\mathbf{Y} \subset \mathrm{Mod}_\alpha(\mathbf{C})$ is the full graded subcategory spanned by the objects $\mathbf{T}(-,X)_{|\mathbf{C}}$, then θ_2 represents a class in Hochschild–Mitchell cohomology,

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Corollary

α-p. gl. dim $\mathbf{T} \leq 1$ *iff* [ARO] *and* [ARM] *hold.*

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Counterexamples to Rosický's theorem

Proposition (Christensen–Keller–Neeman'01 for $\alpha = \aleph_0$)

If R is an α-coherent ring, $\mathbf{T} = D(R)$ and $\mathbf{C} = \alpha$-compact complexes, then for any $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Ab}$ in $\mathrm{Mod}_\alpha(\mathbf{C})$

$$\alpha\text{-p. gl. dim}\,\mathbf{T} \geq \alpha\text{-p. gl. dim}\,R.$$

α-Purity in a the category of R-modules is the homological algebra arising from pretending that R-modules with $< \alpha$ generators and relations are projective.

Theorem

[Braun–Göbel'10] α-pure global dim $\mathbb{Z} > 1$ *for any* $\alpha > \aleph_0$.
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Rosický's theorem is false for $D(\mathbb{Z})$ and $D(\mathbb{C}[x, y])$.

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Transfinite Adams' representability for objects

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More precisely:

- [ARM] *does not hold for* $D(\mathbb{Z})$ *and* $\alpha > \aleph_0$.
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Question (Transfinite [ARO])

If **T** is a well generated triangulated category and **C** denotes the category of α-compact objects, is it true that for big enough regular cardinals α any cohomological functor $H \colon \mathbf{C}^{op} \to \mathbf{Ab}$ is of the form $H \cong \mathbf{T}(-, X)_{|\mathbf{C}}$ for some object X in **T**?

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$\aleph_1$-Adams' representability for objects

Proposition

If the cardinal of the category $\mathbf{C}$ *of* $\aleph_1$*-compact objects is* $\leq \aleph_1$ *then any cohomological functor* $H\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Ab}$ *is* $H \cong \mathbf{T}(-, X)_{|\mathbf{C}}$.

Example

Provided $2^{\aleph_0} = \aleph_1$ *(continuum hypothesis):*

- $D(R)$ *if* card $R \leq \aleph_1$, *e.g.* $\mathbb{C}[x, y]$.
- *Stable homotopy category.*
- $K(\mathrm{Proj}(R))$ *if* card $R \leq \aleph_1$.
- $D(\mathrm{Sh}(M))$ *if* M *is a connected open manifold.*

If $2^{\aleph_0} = \aleph_1$ *and* $2^{\aleph_1} = \aleph_2$*:*

- *The stable motivic homotopy category over a finite-dimensional noetherian scheme with a cover by* Spec(*rings of* card $\leq \aleph_1$).

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What are the obstructions to the representability of an object F in $\mathrm{Mod}_\alpha(\mathbf{C})$ fitting into an extension as follows?

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It represents an element in $\mathrm{Ext}^{1,0}_{\mathbf{C}}(\mathbf{T}(-,X)_{|\mathbf{C}}, \mathbf{T}(-,Y)_{|\mathbf{C}})$.

There is a conditionally convergent Adams spectral sequence [Christensen'98]

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Corollary (Christensen–Keller–Neeman'01 for $\alpha = \aleph_0$)

If R is a hereditary ring, $\mathbf{T} = D(R)$ and $\mathbf{C}$ is the category of α-compact complexes, the following statements are equivalent:

- *Any cohomological functor $H\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Ab}$ is $H \cong \mathbf{T}(-, X)_{|\mathbf{C}}$.*
- α-p. gl. dim $R \leq 2$.

Question (Transfinite [ARO] for $D(R)$ with R hereditary)

Is there any hereditary ring R with α-pure projective dimension > 2 for $\alpha > \aleph_0$?

Otherwise, for any R-module M the kernel of

$$\text{induced by inclusions: } \bigoplus_{\substack{N \subset M \\ N\ \alpha\text{-generated}}} N \twoheadrightarrow M$$

would be a direct summand of some $\bigoplus_{i \in I} P_i$ with P_i α-generated.

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Obstructions to Adams representability

Fernando Muro

Universidad de Sevilla

(based on joint work with O. Raventos, from U. Barcelona)

Triangulated categories and applications
Banff, June 12–17, 2011

Grading

We can regard **T** as a graded category with graded morphism sets

$$\mathbf{T}^*(X, Y) = \bigoplus_{n\in\mathbb{Z}} \mathbf{T}(X, \Sigma^n Y).$$

Since **C** is closed under Σ and Σ^{-1}, the suspension functor extends to an exact equivalence

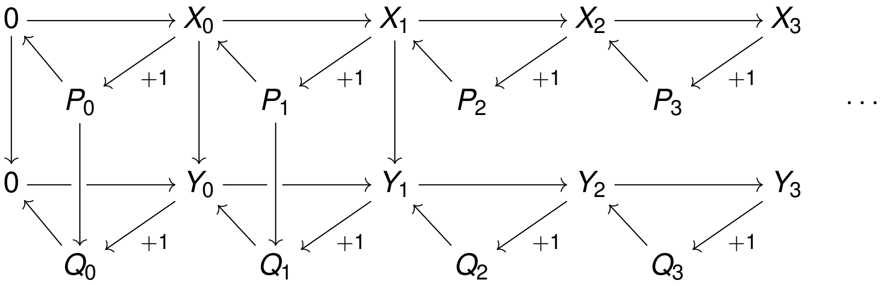
$$\Sigma\colon \operatorname{Mod}_\alpha(\mathbf{C}) \xrightarrow{\sim} \operatorname{Mod}_\alpha(\mathbf{C})$$

which induces a graded abelian category structure in $\operatorname{Mod}_\alpha(\mathbf{C})$ in the same way.

back

Definition of the first obstruction $\theta_2(\tau)$

Given $\tau\colon \mathbf{T}(-,X)_{|\mathbf{C}} \to \mathbf{T}(-,Y)_{|\mathbf{C}}$, extend it to a 1-truncated morphism between good Postnikov resolutions of $\mathbf{T}(-,X)_{|\mathbf{C}}$ and $\mathbf{T}(-,Y)_{|\mathbf{C}}$



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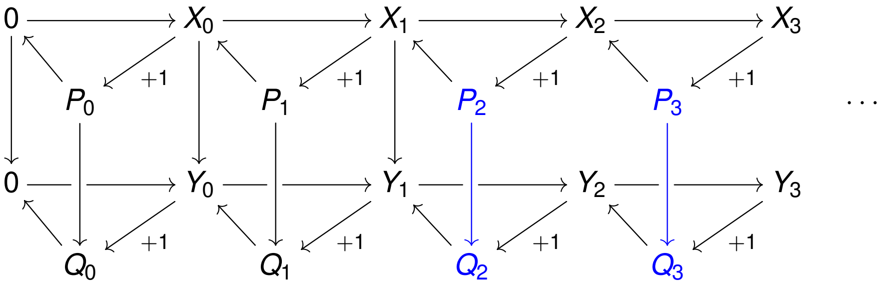

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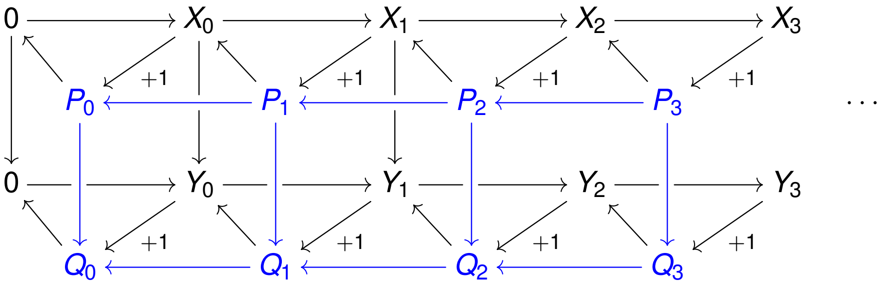

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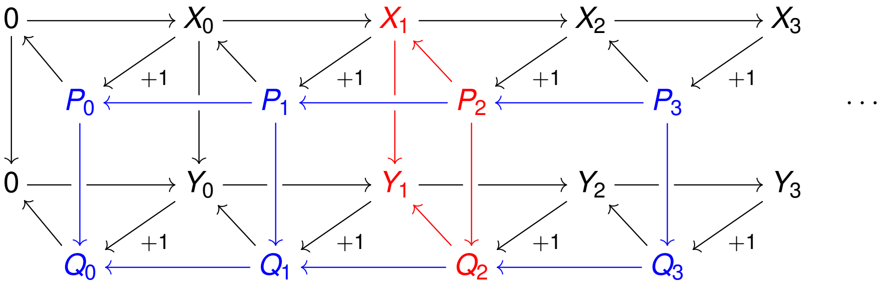

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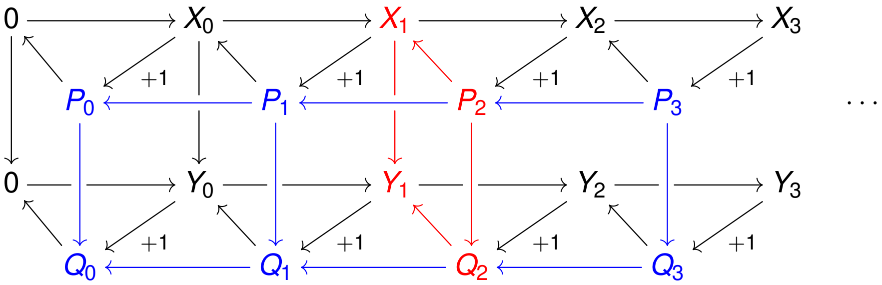

Need not commute! The lack of commutativity of the red square is measured by a morphism

$$P_2 \xrightarrow{-1} \operatorname{hocolim} Y_n = Y$$

which represents $\theta_2^{X,Y}(\tau) \in \operatorname{Ext}^{2,-1}_{\mathbf{C}}(\mathbf{T}(-,X)_{|\mathbf{C}}, \mathbf{T}(-,Y)_{|\mathbf{C}})$. back

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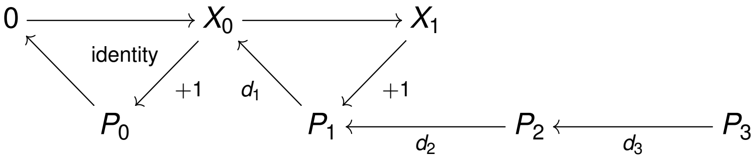

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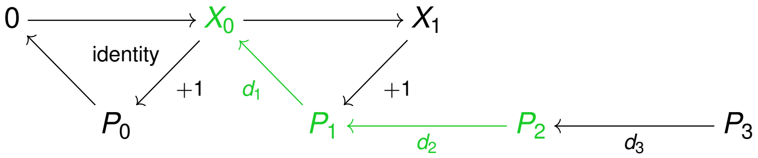

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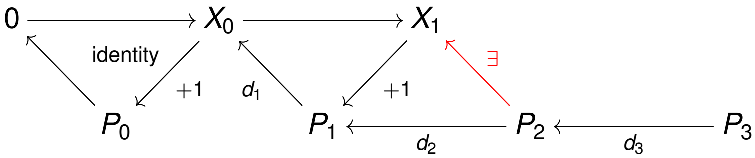

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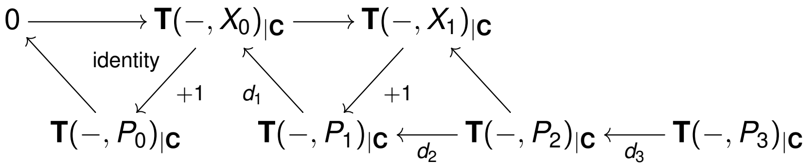

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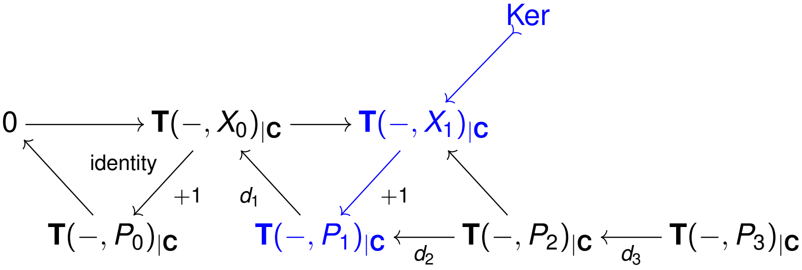

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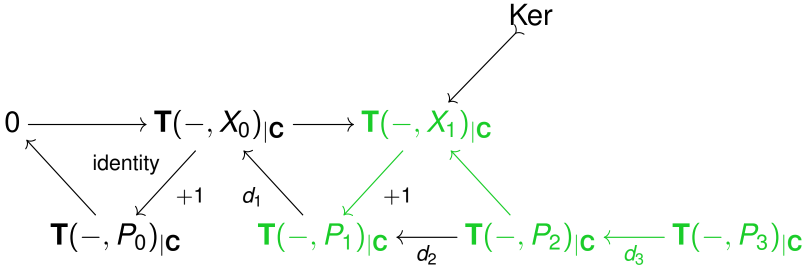

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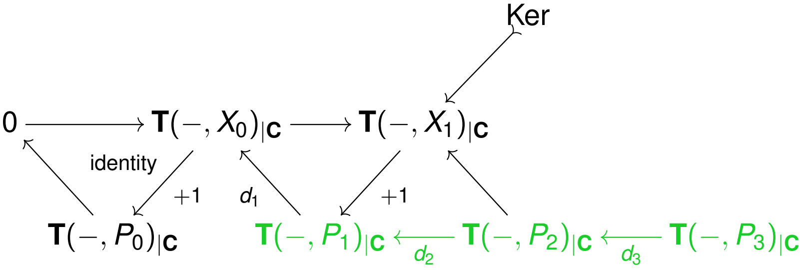

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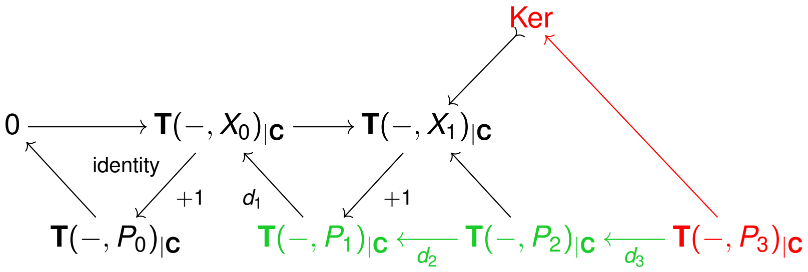

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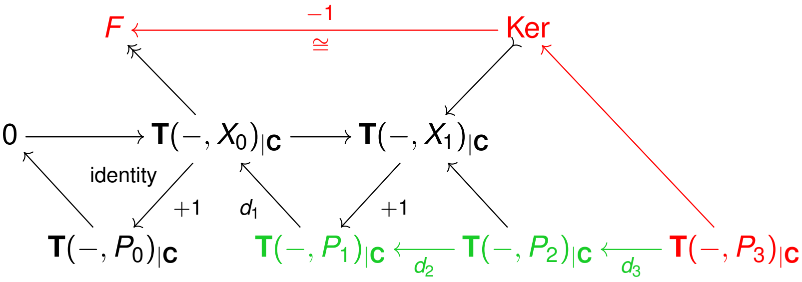

The red composite represents $\kappa_3(F) \in \mathrm{Ext}^{3,-1}_{\mathbf{C}}(F, F)$. back