The triangulated Auslander–Iyama correspondence II

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Main theorem

Theorem (Jasso–M.'22)

Let k be a perfect field and $d \geq 1$. There is a bijection between equivalence classes of pairs:

1. $(\mathcal{T}, c)$ where:
   a) $\mathcal{T}$ a small algebraic triangulated category with finite-dimensional Hom's and split idempotents.
   b) c a basic $d\mathbb{Z}$-cluster tilting object.
2. $(\Lambda, [\sigma])$ where:
   a) Λ a basic finite-dimensional self-injective twisted $(d+2)$-periodic algebra.
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The bijection is given by $\Lambda = \mathcal{T}(c, c)$ and ${}_1\Lambda_\sigma = \mathcal{T}(c[d], c)$.[a]

These triangulated categories admit a unique DG enhancement.

[a] As objects, $c[d] = c$ but $[d]$ does not act like the identity on $\Lambda = \mathcal{T}(c, c)$.

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What structure does $\mathcal{C} = \mathsf{add}(c)$ inherit from $\mathcal{T}$?

$(d+2)$-angulated categories

Definition (Geiss, Keller, and Oppermann 2013)

A $(d+2)$-angulated category is a category $\mathcal{C}$ equipped with a self-equivalence

$$\Sigma\colon \mathcal{C} \xrightarrow{\sim} \mathcal{C},$$

called suspension, and a class of diagrams, called exact $(d+2)$-angles,

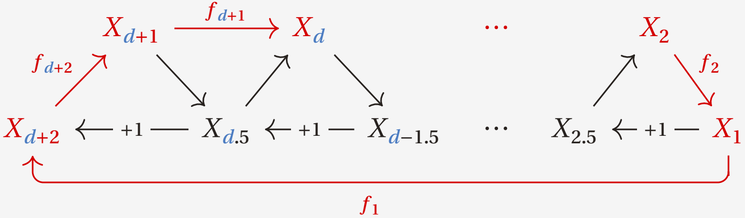
$$x_{d+2} \xrightarrow{f_{d+2}} x_{d+1} \xrightarrow{f_{d+1}} \cdots \xrightarrow{f_2} x_1 \xrightarrow{f_1} \Sigma x_{d+2},$$

satisfying axioms similar to those of triangulated categories (which is the case $d = 1$).

GKO $(d + 2)$-angulated categories

Theorem (Geiss, Keller, and Oppermann 2013)

If $\mathcal{C} \subset \mathcal{T}$ is a $d\mathbb{Z}$-cluster tilting subcategory then $(\mathcal{C}, [d])$ is $(d + 2)$-angulated with exact $(d + 2)$-angles



Properties of $(d + 2)$-angulated categories

Proposition (Heller 1968; Geiss, Keller, and Oppermann 2013)

If $\mathcal{C}$ is $(d + 2)$-angulated then $\mathsf{mod}(\mathcal{C})$ is a Frobenius abelian category.

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If $\mathcal{C}$ is $(d+2)$-angulated then $\mathsf{mod}(\mathcal{C})$ is a Frobenius abelian category.

If $\mathcal{C} = \mathsf{add}(c) \subset \mathcal{T}$ and $\Lambda = \mathcal{T}(c, c)$ then

$$\mathcal{C} \simeq \mathsf{proj}(\Lambda), \qquad \mathsf{mod}(\mathcal{C}) \simeq \mathsf{mod}(\Lambda).$$

Corollary

If $c \in \mathcal{T}$ is basic $d\mathbb{Z}$-cluster tilting then Λ is self-injective.

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Corollary
If $c \in \mathcal{T}$ is basic $d\mathbb{Z}$-cluster tilting then Λ is self-injective.

Proposition (Hanihara 2020; Jasso–M. 22)
If $c \in \mathcal{T}$ is basic $d\mathbb{Z}$-cluster tilting then Λ is twisted $(d+2)$-periodic.

AL $(d+2)$-angulated categories

Theorem (Amiot 2007; Lin 2019)

If Λ is a basic finite-dimensional self-injective twisted $(d+2)$-periodic algebra w.r.t. an automorphism σ and

$$ {}_1\Lambda_\sigma \hookrightarrow P_{d+2} \to P_{d+1} \to \cdots \to P_2 \to P_1 \twoheadrightarrow \Lambda $$

is an extension with projective-injective middle Λ-bimodules P_i, in particular $\Omega^{d+2}_{\Lambda^e}(\Lambda) \cong {}_1\Lambda_\sigma$ in $\underline{\mathsf{mod}}(\Lambda^e)$, then

$$ \mathsf{proj}(\Lambda) $$

is $(d+2)$-angulated with desuspension functor

$$ \sigma^*\colon \mathsf{proj}(\Lambda) \longrightarrow \mathsf{proj}(\Lambda). $$

This applies to all basic finite-dimensional self-injective algebras of finite representation type (Dugas 2010).

AL $(d+2)$-angulated categories

A $(d+2)$-angle in $\operatorname{proj}(\Lambda)$

$$(X_{d+2})_\sigma \longrightarrow X_{d+1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \xrightarrow{f} X_{d+2},$$

is exact if it satisfies the two following conditions:

1. The following extended sequence is exact

$$(X_1)_\sigma \xrightarrow{f} (X_{d+2})_\sigma \longrightarrow X_{d+1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \xrightarrow{f} X_{d+2}.$$

2. The induced extension in $\operatorname{mod}(\Lambda)$ with $M = \operatorname{im} f$

$$M_\sigma \overset{i}{\hookrightarrow} (X_{d+2})_\sigma \longrightarrow X_{d+1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \overset{p}{\twoheadrightarrow} M$$

and $f = ip$ is equivalent to the following one in $\operatorname{Ext}_\Lambda^{d+2}(M, M_\sigma)$

$$M \otimes_\Lambda ({}_1\Lambda_\sigma \hookrightarrow P_{d+2} \to P_{d+1} \to \cdots \to P_2 \to P_1 \twoheadrightarrow \Lambda).$$

AL $(d+2)$-angulated categories

Proposition (Jasso–M.'22)

Let Λ be a basic finite-dimensional self-injective twisted $(d+2)$-periodic algebra w.r.t. an automorphism σ. Up to equivalence, the AL $(d+2)$-angulated structure on $\mathsf{proj}(\Lambda)$ with desuspension σ^* is independent of the choice of an extension with projective-injective middle terms.

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Proof (idea)

$$\begin{array}{ccccccccccccc}
{}_1\Lambda_\sigma & \hookrightarrow & P_{d+2} & \longrightarrow & P_{d+1} & \longrightarrow & \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \twoheadrightarrow & \Lambda \\
{\scriptstyle u\cdot -}\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \parallel \\
{}_1\Lambda_\sigma & \hookrightarrow & Q_{d+2} & \longrightarrow & Q_{d+1} & \longrightarrow & \cdots & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \twoheadrightarrow & \Lambda
\end{array}$$

where $u \in Z(\Lambda) \twoheadrightarrow \underline{Z}(\Lambda)$ maps to a unit, and $Z(\Lambda)^\times \twoheadrightarrow \underline{Z}(\Lambda)^\times$ is also surjective since $\dim \Lambda < \infty$. $\square$

$(d+2)$-angulated categories recap

basic $d\mathbb{Z}$-cluster tilting $c \in \mathcal{T}$ triangulated

$$\Downarrow$$

$\mathcal{C} = \mathsf{add}(c)$ is $(d+2)$-angulated

$$\Updownarrow$$

$\Lambda = \mathcal{T}(c, c)$ is self-injective twisted $(d+2)$-periodic

Warning!

$(d+2)$-angulated categories

If $c \in \mathcal{T}$ is a basic $d\mathbb{Z}$-cluster tilting object and $\Lambda = \mathcal{T}(c, c)$ then

$$\mathcal{C} = \mathsf{add}(c) \simeq \mathsf{proj}(\Lambda)$$

carries two $(d+2)$-angulated structures:

1. The GKO structure since $\mathcal{C} \subset \mathcal{T}$ is $d\mathbb{Z}$-cluster tilting.
2. The AL structure for Λ is self-injective twisted $(d+2)$-periodic.

These structures need not be equivalent!

Main theorem

Theorem (Jasso–M.'22)

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These triangulated categories admit a unique DG enhancement.

Enhancements

A triangulated category $\mathcal{T}$ is algebraic (Keller 2007) if

$$\mathcal{T} \simeq \underbrace{\mathsf{D}^c(\mathcal{A})}_{\text{Perfect derived category}}$$

for some DG category $\mathcal{A}$, which is then called enhancement.

A DG functor $f\colon \mathcal{A} \to \mathcal{B}$ is a Morita equivalence if

$$\mathbb{L}f_*\colon \mathsf{D}^c(\mathcal{A}) \xrightarrow{\sim} \mathsf{D}^c(\mathcal{B})$$

is an equivalence.

An algebraic triangulated category has a unique enhancement if any two enhancements are Morita equivalent.

A DG functor $f\colon \mathcal{A} \to \mathcal{B}$ is a quasi-equivalence if

$$H^0(f)\colon H^0(\mathcal{A}) \xrightarrow{\sim} H^0(\mathcal{B}), \quad f\colon \mathcal{A}(x,y) \xrightarrow{\sim} \mathcal{B}(f(x), f(y)), \quad x, y \in \mathcal{A}.$$

Quasi-equivalences are Morita equivalences.

Enhancements

Given a DG category $\mathcal{A}$ we have a Yoneda embedding

$$\begin{aligned} H^0(\mathcal{A}) &\hookrightarrow \mathsf{D}^c(\mathcal{A}), \\ x &\mapsto \mathcal{A}(-, x). \end{aligned}$$

A DG category $\mathcal{A}$ is pre-triangulated (Bondal and Kapranov 1991) if the Yoneda embedding is an equivalence

$$H^0(\mathcal{A}) \xhookrightarrow{\sim} \mathsf{D}^c(\mathcal{A}).$$

Morita equivalences between pre-triangulated categories are quasi-equivalences.

Any DG category embeds in a pre-triangulated one through a Morita equivalence

$$\mathcal{A} \hookrightarrow \operatorname{tri} \mathcal{A}.$$

The cohomology of an enhancement

If $\mathcal{A}$ is a pre-triangulated enhancement of $\mathcal{T} = H^0(\mathcal{A}) \simeq \mathsf{D}^c(\mathcal{A})$

$$H^{n}(\mathcal{A}(x,y)) \cong \mathcal{T}(x, y[n]) = \mathcal{T}^{n}(x,y), \qquad x, y \in \mathcal{T}, \quad n \in \mathbb{Z}.$$

If $\mathcal{T} = \langle c \rangle$ then

$$\mathcal{A}(c,c) \hookrightarrow \mathcal{A}$$

is a Morita equivalence.

If $c \in \mathcal{T}$ is basic $d\mathbb{Z}$-cluster tilting

$$H^*(\mathcal{A}(c,c)) \cong \mathcal{T}^*(c,c) = \bigoplus_{i\in\mathbb{Z}} \underbrace{\mathcal{T}^*(c, c[di])}_{\text{degree } di}$$

is d-sparse.

The cohomology of an enhancement

If $\Lambda = \mathcal{T}(c, c)$ and $[\sigma] \in \mathrm{Out}(\Lambda)$ is such that

$$\mathcal{T}(c[d], c) \cong {}_1\Lambda_\sigma$$

as a Λ-bimodule then

$$H^*(\mathcal{A}(c, c)) \cong \bigoplus_{i \in \mathbb{Z}} \underbrace{{}_{\sigma^i}\Lambda_1}_{\text{degree } di} = \frac{\Lambda\langle t^{\pm 1}\rangle}{(t\lambda - \sigma(\lambda)t)_{\lambda \in \Lambda}} =: \Lambda(\sigma, d), \quad |t| = -d.$$

Recovering the enhancement from the cohomology

The DG algebra $\mathcal{A}(c, c)$ is determined by a minimal A_∞-algebra structure on $\Lambda(\sigma, d)$, given by operations (Stasheff 1963)

$$m_n \colon \Lambda(\sigma, d) \otimes \overset{n}{\cdots} \otimes \Lambda(\sigma, d) \longrightarrow \Lambda(\sigma, d), \qquad |m_n| = 2 - n, \quad n \geq 1,$$

satisfying equations:

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- $m_{d+2} \in \mathrm{C}^{d+2,-d}(\Lambda(\sigma, d))$ is a Hochschild cocycle. Its class

$$\{m_{d+2}\} \in \mathrm{HH}^{d+2,-d}(\Lambda(\sigma, d))$$

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$$\begin{aligned} j^*\colon \mathrm{HH}^{d+2,-d}(\Lambda(\sigma,d),\Lambda(\sigma,d)) &\longrightarrow \mathrm{HH}^{d+2,-d}(\Lambda,\Lambda(\sigma,d)),\\ \{m_{d+2}\} &\longmapsto \underbrace{j^*\{m_{d+2}\}}_{\text{Restricted universal Massey product}}. \end{aligned}$$

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The target is

$$\begin{aligned} \mathsf{HH}^{d+2,-d}(\Lambda,\Lambda(\sigma,d)) &= \mathsf{HH}^{d+2}(\Lambda,\Lambda(\sigma,d)^{-d}) \\ &= \mathsf{HH}^{d+2}(\Lambda, {}_{\sigma^{-1}}\Lambda_1) \\ &\cong \mathsf{HH}^{d+2}(\Lambda, {}_1\Lambda_\sigma) \\ &= \mathsf{Ext}^{d+2}_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma). \end{aligned}$$

The universal Massey product

Consider a representative of $j^*\{m_{d+2}\} \in \mathsf{Ext}^{d+2}_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma)$

$$ {}_1\Lambda_\sigma \hookrightarrow P_{d+2} \to P_{d+1} \to \cdots \to P_2 \to P_1 \twoheadrightarrow \Lambda $$

with P_i projective-injective except for P_{d+2}, possibly.

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Let $\mathcal{T}$ be an algebraic triangulated category and $c \in \mathcal{T}$ a basic $d\mathbb{Z}$-cluster tilting object. Then P_{d+2} is also projective-injective and the GKO $(d+2)$-angulated structure on $\mathcal{C} = \mathsf{add}(c) \simeq \mathsf{proj}(\Lambda)$ coincides with the AL one.

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Theorem (Jasso–M.'22)

Let $\mathcal{T}$ be an algebraic triangulated category and $c \in \mathcal{T}$ a basic $d\mathbb{Z}$-cluster tilting object. Then P_{d+2} is also projective-injective and the GKO $(d+2)$-angulated structure on $\mathcal{C} = \mathsf{add}(c) \simeq \mathsf{proj}(\Lambda)$ coincides with the AL one.

The property of P_{d+2} being also projective-injective is equivalent to $j^*\{m_{d+2}\}$ being a unit in Holchschild–Tate cohomology

$$ \underline{\mathsf{HH}}^{\bullet,*}(\Lambda, \Lambda(\sigma, d)) = \underline{\mathsf{Ext}}^{\bullet,*}_{\Lambda^e}(\Lambda, \Lambda(\sigma, d)). $$

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Theorem (Kadeishvili 1988)

$\mathrm{HH}^{n+2,-n}(A) = 0$ for $n > 0 \Rightarrow A$ is intrinsically formal.

If $H^*(B) \cong A$ is d-sparse and $\{m^B_{d+2}\} \neq 0 \in \mathrm{HH}^{d+2,-d}(A)$ then A is not intrinsically formal. In particular $\Lambda(\sigma, d)$ is not intrinsically formal if Λ is not separable.

A separable example

Let $d \geq 2$ be even. The algebraic triangulated category

$$\mathcal{T} = \mathsf{D}^c(k[t^{\pm 1}]) \simeq \mathsf{mod}(k) \times \overset{d}{\cdots} \times \mathsf{mod}(k), \qquad |t| = -d,$$

has a basic $d\mathbb{Z}$-cluster tilting object

$$c = k[t^{\pm 1}] \mapsto (k, 0, \dots, 0)$$

with (intrinsically formal graded) endomorphism algebra

$$\mathcal{T}(c, c) = \Lambda = k, \qquad \mathcal{T}^*(c, c) = \Lambda(\sigma, d) = k[t^{\pm 1}],$$

by Kadeishvili 1988, since

$$\mathsf{HH}^{\bullet,*}(k[t^{\pm 1}]) = k[t^{\pm 1}, \delta]/(\delta^2)$$

with

$$\delta \in \mathsf{HH}^{1,0}(k[t^{\pm 1}])$$

the fractional Euler class, defined by $\delta_{|_\Lambda} = 0$ and $\delta(t) = -t$.

Massey formality

A d-sparse Massey algebra (A, m) is a d-sparse graded algebra A and

$$m \in \mathrm{HH}^{d+2,-d}(A), \qquad \frac{[m, m]}{2} = 0.$$

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We say that (A, m) is Massey formal if, given DG algebras B_1, B_2,

$$\left.\begin{aligned} H^*(B_i) &\cong A \\ \{m^{B_i}_{d+2}\} &\mapsto m \end{aligned}\right\} \Longrightarrow B_1 \simeq B_2.$$

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We say that (A, m) is Massey formal if, given DG algebras B_1, B_2,

$$\left.\begin{aligned} H^*(B_i) &\cong A \\ \{m_{d+2}^{B_i}\} &\mapsto m \end{aligned}\right\} \Longrightarrow B_1 \simeq B_2.$$

The Hochschild cohomology of a d-sparse Massey algebra $\mathrm{HH}^{\bullet,*}(A, m)$ is the cohomology of the complex

$$(\mathrm{HH}^{\bullet,*}(A), [m, -]).$$

Massey formality

A d-sparse Massey algebra (A, m) is a d-sparse graded algebra A and

$$m \in \mathrm{HH}^{d+2,-d}(A), \qquad \frac{[m, m]}{2} = 0.$$

We say that (A, m) is Massey formal if, given DG algebras B_1, B_2,

$$\left.\begin{array}{r} H^*(B_i) \cong A \\ \{m_{d+2}^{B_i}\} \mapsto m \end{array}\right\} \Longrightarrow B_1 \simeq B_2.$$

The Hochschild cohomology of a d-sparse Massey algebra $\mathrm{HH}^{\bullet,*}(A, m)$ is the cohomology of the complex

$$(\mathrm{HH}^{\bullet,*}(A), [m, -]).$$

Theorem (Jasso–M.'22)

$\mathrm{HH}^{n+2,-n}(A, m) = 0$ for $n > d \Rightarrow (A, m)$ is Massey formal.

Uniqueness

Theorem (Jasso–M.'22)

Let $\mathcal{T}$ be an algebraic triangulated category with basic $d\mathbb{Z}$-cluster tilting object $c \in \mathcal{T}$. If $\Lambda = \mathcal{T}(c, c)$ and $\mathcal{T}(c[d], c) \cong {}_1\Lambda_\sigma$ then $(\Lambda(\sigma, d), \{m_{d+2}\})$ is Massey formal. In particular $\mathcal{T}$ has a unique enhancement.

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1. $j^*\{m_{d+2}\} \in \mathrm{HH}^{d+2,-d}(\Lambda, \Lambda(\sigma, d))$ is a unit in $\underline{\mathrm{HH}}^{\bullet,*}(\Lambda, \Lambda(\sigma, d))$.

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3. The product with $j^*\{m_{d+2}\}$ in $\mathrm{HH}^{\bullet,*}(\Lambda, \Lambda(\sigma, d))$ is an isomorphism for $\bullet > 0$ and surjective for $\bullet = 0$.
4. We have a square-zero extension

$$\mathrm{HH}^{\bullet-1,*}(\Lambda, \Lambda(\sigma, d))_{\langle\sigma\rangle} \hookrightarrow \mathrm{HH}^{\bullet,*}(\Lambda(\sigma, d)) \xrightarrow{j^*} \mathrm{HH}^{\bullet,*}(\Lambda, \Lambda(\sigma, d))^{\langle\sigma\rangle}.$$

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6. The product with $\{m_{d+2}\}$ in $\mathrm{HH}^{\bullet,*}(\Lambda(\sigma, d))$ is nullhomotopic:

$$\{m_{d+2}\} \cdot x = [\{m_{d+2}\}, \delta \cdot x] + \delta \cdot [\{m_{d+2}\}, x].$$

To see it like a null-homotopy:

$$f(x) = \partial h(x) + h\partial(x),$$
$$f(x) = \{m_{d+2}\} \cdot x,$$
$$\partial(x) = [\{m_{d+2}\}, x],$$
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7. $\mathrm{HH}^{\bullet,*}(\Lambda(\sigma, d)) = 0$ for $\bullet > d$.

□

Main theorem

Theorem (Jasso–M.'22)

Let k be a perfect field and $d \geq 1$. There is a bijection between equivalence classes of pairs:

1. $(\mathcal{T}, c)$ where:
   a) $\mathcal{T}$ an algebraic triangulated category with finite-dimensional Hom's and split idempotents.
   b) c a basic $d\mathbb{Z}$-cluster tilting object.
2. $(\Lambda, [\sigma])$ where:
   a) Λ a basic finite-dimensional self-injective twisted $(d+2)$-periodic algebra.
   b) $[\sigma] \in \mathsf{Out}(\Lambda)$ such that $\Omega^{d+2}_{\Lambda^e}(\Lambda) \cong {}_1\Lambda_\sigma$ in $\underline{\mathsf{mod}}(\Lambda^e)$.

The bijection is given by $\Lambda = \mathcal{T}(c, c)$ and ${}_1\Lambda_\sigma = \mathcal{T}(c[d], c)$.

These triangulated categories admit a unique DG enhancement.

Surjectivity

Given a basic self-injective algebra Λ, an automorphism σ and an extension

$$\eta\colon \qquad {}_1\Lambda_\sigma \hookrightarrow P_{d+2} \to P_{d+1} \to \cdots \to P_2 \to P_1 \twoheadrightarrow \Lambda$$

with projective-injective middle terms, we construct a DG algebra B such that $\mathcal{T} = \mathsf{D}^c(B)$ has basic $d\mathbb{Z}$-cluster tilting object $c = B$ with

$$\mathcal{T}(c, c) \cong \Lambda, \qquad\qquad \mathcal{T}(c[d], c) \cong {}_1\Lambda_\sigma,$$

in the following way.

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in the following way.

It suffices to have

$$H^*(B) \cong \Lambda(\sigma, d), \qquad j^*\{m_{d+2}\} \mapsto \{\eta\} \in \mathsf{HH}^{d+2,-d}(\Lambda, \Lambda(\sigma, d)) \cong \mathsf{Ext}^{d+2}_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma).$$

Surjectivity

Theorem (Jasso–M.'22)

There exists a unique $m \in \mathsf{HH}^{d+2,-d}(\Lambda(\sigma, d), \Lambda(\sigma, d))$ such that

$$j^*(m) = \{\eta\}, \qquad \frac{[m,m]}{2} = 0.$$

Surjectivity

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Theorem (Jasso–M.'22)

Given a d-sparse Massey algebra (A, m), if $\mathrm{HH}^{n+1,-n}(A, m) = 0$ for $n > 2d + 3$ then there exists a DG algebra B with

$$H^*(B) \cong A, \qquad \{m_{d+2}\} \mapsto m.$$

Thanks for your attention! ☺

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