Kadeishvili's formality theorem and universal Massey products

Higher Structures in Rome

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DG algebra $A \longmapsto (H^*A, m_3, m_4, \dots)$ A_∞***-algebra*** (minimal model),

$$m_n : H^*A \otimes \overset{n}{\cdots} \otimes H^*A \longrightarrow H^*A, \qquad |m_n| = 2 - n, \qquad n \geq 3.$$

Kadeishvili's intrinsic formality theorem

Theorem (Kadeishvili 1988)

Let A be a DG algebra. If

$$\mathrm{HH}^{n+2,-n}(H^*A) = 0, \qquad n \geq 1,$$

then A is formal.

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Corollary (Equivalent form)

Let A and B be DG algebras with

$$H^*A \cong H^*B$$

as graded algebras. If

$$\mathrm{HH}^{n+2,-n}(H^*A) = 0, \qquad n \geq 1,$$

then

$$B \simeq A.$$

The universal Massey product

Consider a minimal A_∞-model of a DG algebra A,

$$(H^*A, m_3, m_4, \dots).$$

Definition

The higher multiplication m_3 is a Hochschild cocycle, and its cohomology class

$$\{m_3\} \in \mathrm{HH}^{3,-1}(H^*A)$$

is the ***universal Massey product (UMP)*** of A.

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The UMP is:

- independent of the choice of minimal model,
- invariant under quasi-isomorphisms,
- the first obstruction to formality,

$$[\{m_3\}, \{m_3\}] = 0.$$

Massey algebras and Hochschild–Massey cohomology

Definition

A ***Massey algebra*** is a pair (H, m), where H is a graded algebra and

$$m \in \mathrm{HH}^{3,-1}(H), \qquad [m, m] = 0.$$

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Definition

The ***Hochschild–Massey complex*** of a Massey algebra (H, m) is

$$\big(\mathrm{HH}^{*,*}(H), [m, -]\big).$$

Its cohomology is the ***Hochschild–Massey cohomology*** of (H, m), denoted by

$$\mathrm{HH}^{*,*}(H, m).$$

The generalized Kadeishvili theorem

Theorem (Jasso and Muro 2023)

Let A and B be DG algebras with

$$H^*A \cong H^*B$$

as graded algebras and, under this identification,

$$\{m_3^A\} = \{m_3^B\}.$$

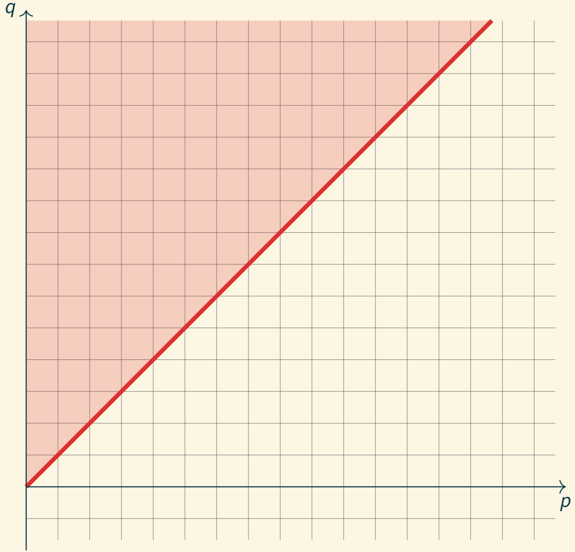
If

$$\mathrm{HH}^{n+2,-n}(H^*A, \{m_3^A\}) = 0, \qquad n > 1,$$

then

$$A \simeq B.$$

Proof of the generalized Kadeishvili theorem



Bousfield and Kan 1972 spectral sequence of the space of minimal A_∞-algebra structures on H^*A, pointed at the minimal model of A,

$$\mathrm{Map}(\mathcal{A}_\infty, \mathcal{E}(H^*A)) = \lim_n \mathrm{Map}(\mathcal{A}_n, \mathcal{E}(H^*A));$$

$$E_2^{pq} \Rightarrow \pi_{q-p} \mathrm{Map}(\mathcal{A}_\infty, \mathcal{E}(H^*A));$$

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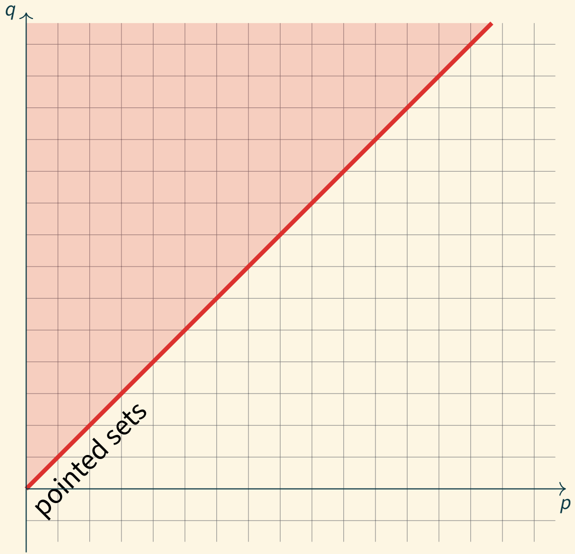

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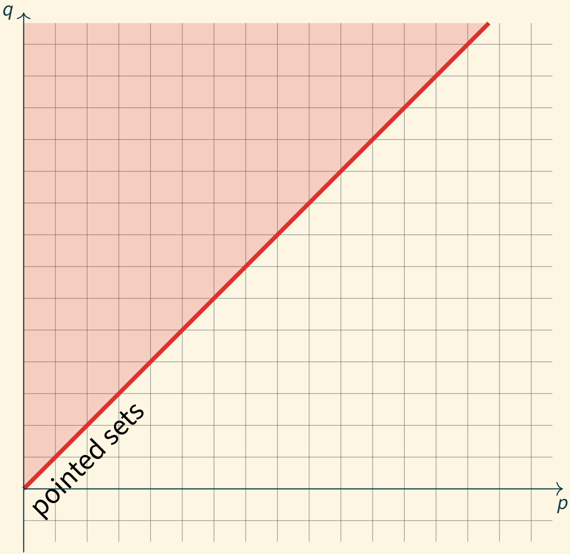

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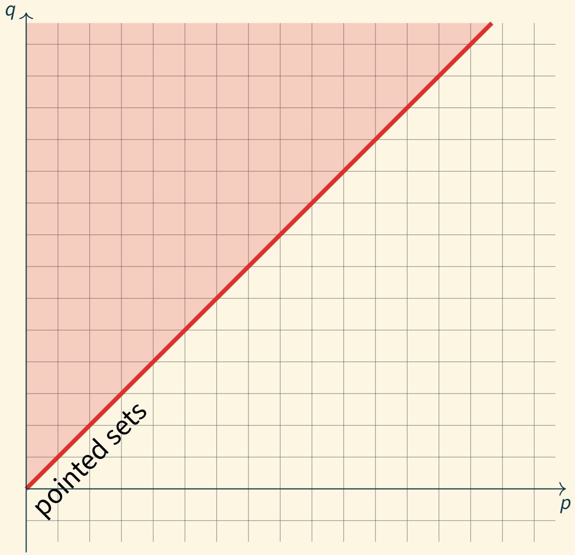

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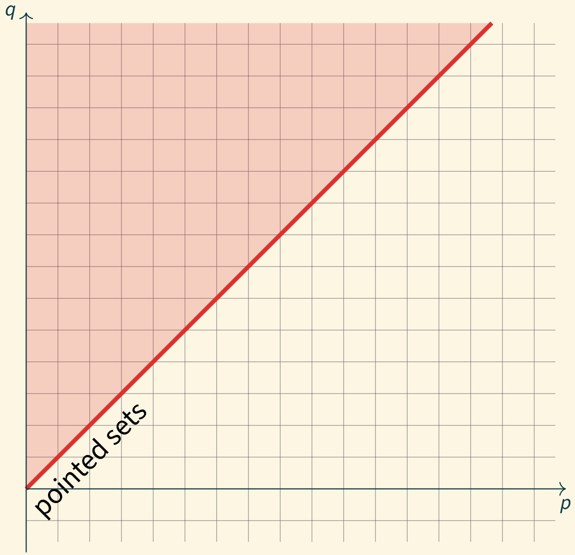

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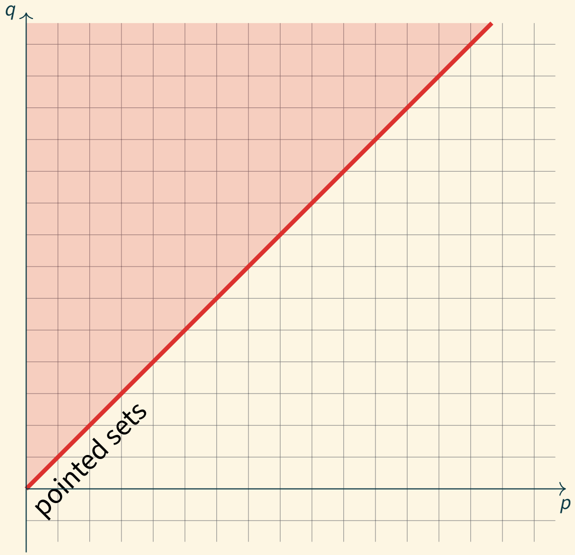

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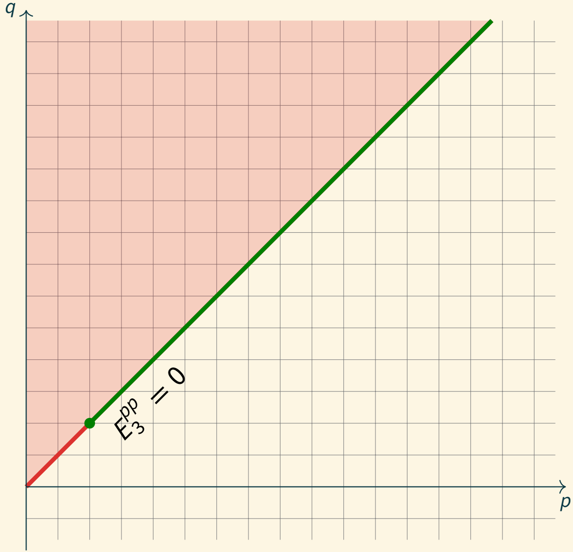

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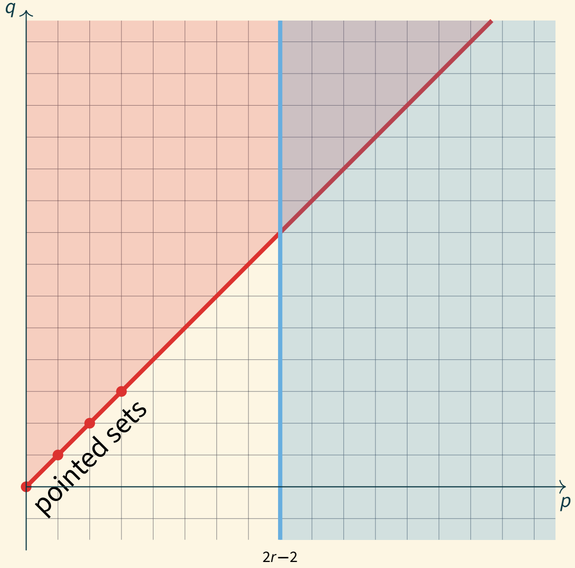

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The derived Auslander correspondence, Muro 2022

The following result is an application of the generalized Kadeishvili theorem.

Theorem (Derived Auslander correspondence)

There is a bijection

$$\left\{\begin{array}{c}\text{DG algebras } A \text{ up to quasi-isomorphism}\\ H^0(A) \text{ basic finite-dimensional}\\ A \text{ an additive generator of } D^c(A)\end{array}\right\} \longrightarrow \left\{\begin{array}{c}(\Lambda, I) \text{ up to isomorphism}\\ \Lambda \text{ basic Frobenius algebra}\\ I \text{ an invertible } \Lambda\text{-bimodule}\\ \text{such that } \Omega^3_{\Lambda^e}(\Lambda) \cong I \text{ stably}\end{array}\right\},$$

$$A \longmapsto (H^0A, H^{-1}A).$$

The graded cohomology algebra

Let A be like in the derived Auslander correspondence. Since $\Lambda = H^0 A$ is basic and finite-dimensional and A is an additive generator of $D^c(A)$,

$$A[1] \cong A, \qquad H^{-1}A = D^c(A)(A[1], A) \cong {}_1\Lambda_\sigma, \qquad \textbf{\textit{twisted bimodule,}}$$

for some $\sigma \in \mathrm{Aut}(\Lambda)$, well-defined up to inner automorphism. Twisted bimodules are the invertible Λ-bimodules.

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Given $\varphi\colon A[1] \cong A$ in $D^c(A)$, the automorphism σ of Λ is defined by

$$\sigma(\lambda) = \varphi \circ \lambda[1] \circ \varphi^{-1}, \qquad \lambda\colon A \to A.$$

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Hence,

$$H^*A \cong \bigoplus_{n\in\mathbb{Z}} D^c(A)(A[-n], A) = \bigoplus_{n\in\mathbb{Z}} {}_1\Lambda_{\sigma^{-n}} = \bigoplus_{n\in\mathbb{Z}} {}_1\Lambda_\sigma^{\otimes -n} = \frac{\Lambda\langle t^{\pm 1}\rangle}{(t\lambda - \sigma(\lambda)t)_{\lambda\in\Lambda}} = \Lambda(\sigma), \; |t| = -1.$$

The restricted universal Massey product

The inclusion of the degree 0 part $j\colon \Lambda \hookrightarrow \Lambda(\sigma)$ induces a morphism

$$\begin{aligned} j^*\colon \ \mathrm{HH}^{3,-1}(\Lambda(\sigma),\Lambda(\sigma)) &\longrightarrow \mathrm{HH}^{3,-1}(\Lambda,\Lambda(\sigma)) = \mathrm{Ext}^3_{\Lambda^e}(\Lambda,{}_1\Lambda_\sigma), \\ \{m_3\} &\longmapsto j^*\{m_3\}, \qquad \textit{\textbf{restricted UMP}}. \end{aligned}$$

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Proposition (Muro 2022)

The rUMP of A is represented by a Yoneda extension

$${}_1\Lambda_\sigma \hookrightarrow \underbrace{P_3 \to P_2 \to P_1}_{\text{projective-injective}} \twoheadrightarrow \Lambda$$

which realizes the stable isomorphism $\Omega^3_{\Lambda^e}\Lambda \cong {}_1\Lambda_\sigma$.

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Since Λ is Frobenius, ***Hochschild–Tate cohomology*** is defined,

$$\underline{\mathrm{HH}}^{*,*}(\Lambda,\Lambda(\sigma)).$$

The result says that $j^*\{m_3\}$ is a ***Hochschild–Tate unit***, so $j^*\{m_3\} \neq 0$ unless Λ is separable.
In particular A is non-formal in most cases since $\{m_3\} \neq 0$.

Recovering the universal Massey product

Proposition (Muro 2022)
Given a Hochschild–Tate unit,

$$\eta \in \mathrm{HH}^{3,-1}(\Lambda, \Lambda(\sigma)),$$

there exists a unique

$$m_\eta \in \mathrm{HH}^{3,-1}(\Lambda(\sigma), \Lambda(\sigma))$$

such that

$$j^*(m_\eta) = \eta, \qquad [m_\eta, m_\eta] = 0.$$

Moreover, if $\rho \in \mathrm{HH}^{3,-1}(\Lambda, \Lambda(\sigma))$ is another Hochschild–Tate unit, there exists a graded algebra automorphism of $\Lambda(\sigma)$ under which $m_\eta = m_\rho$.

Therefore, the Massey algebra of A is completely determined by the basic Frobenius algebra $\Lambda = H^0 A$ and the invertible bimodule ${}_1\Lambda_\sigma = H^{-1}A$ such that $\Omega^3_{\Lambda^e}\Lambda \cong {}_1\Lambda_\sigma$ stably.

Why Hochschild–Massey cohomology vanishes

The following result ensures that we can apply the generalized Kadeishvili theorem to prove the injectivity of the derived Auslander correspondence.

Proposition

Given

$$m \in \mathrm{HH}^{3,-1}(\Lambda(\sigma), \Lambda(\sigma))$$

such that

$$j^* m \in \mathrm{HH}^{3,-1}(\Lambda, \Lambda(\sigma))$$

is a Hochschild–Tate unit,

$$\mathrm{HH}^{n+2,*}(\Lambda(\sigma), m) = 0, \qquad n > 1.$$

Why Hochschild–Massey cohomology vanishes

The ***Euler class***

$$\delta \in \mathrm{HH}^{1,0}(\Lambda(\sigma)), \qquad [\delta, x] = q \cdot x, \quad x \in \mathrm{HH}^{*,q}(\Lambda(\sigma))$$

satisfies

$$m \cdot x = [m, \delta \cdot x] + \delta \cdot [m, x], \qquad \textbf{\textit{Gerstenhaber relation}}.$$

Hence, $x \mapsto \delta \cdot x$ is a null-homotopy for $x \mapsto m \cdot x$ in the Hochschild–Massey complex,

$$(\mathrm{HH}^{*,*}(\Lambda(\sigma)), [m, -]).$$

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Moreover, $x \mapsto m \cdot x$ is mostly an isomorphism since $j^* m$ is a Hochschild–Tate unit and there is a long exact sequence,

$$\cdots \to \mathrm{HH}^{p-1,q}(\Lambda, \Lambda(\sigma)) \to \mathrm{HH}^{p,q}(\Lambda(\sigma), \Lambda(\sigma)) \xrightarrow{j^*} \mathrm{HH}^{p,q}(\Lambda, \Lambda(\sigma)) \xrightarrow{1-\mathrm{conj}} \mathrm{HH}^{p,q}(\Lambda, \Lambda(\sigma)) \to \cdots.$$

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Therefore, $\mathrm{HH}^{*,*}(\Lambda(\sigma), m)$ vanishes around the range where $x \mapsto m \cdot x$ is a null-homotopic isomorphism.

The universal Massey product of length $d+2$

A minimal A_∞-model of a DG algebra A with H^*A ***concentrated in degrees*** $d\mathbb{Z}$ for some $d \geq 1$ has many vanishing higher operations for degree reasons. We can denote it by simply indicating the possibly non-trivial ones,

$$(H^*A, m_{d+2}, m_{2d+2}, \dots).$$

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Definition

The higher multiplication m_{d+2} is a Hochschild cocycle, and its cohomology class

$$\{m_{d+2}\} \in \mathrm{HH}^{d+2,-d}(H^*A)$$

is the ***universal Massey product of length*** $d + 2$ of A.

The UMP of length $d + 2$ satisfies

$$[\{m_{d+2}\}, \{m_{d+2}\}] = 0.$$

d-sparse Massey algebras and Hochschild–Massey cohomology

Definition

A ***d-sparse Massey algebra*** is a pair (H, m), where H is a graded algebra concentrated in degrees $d\mathbb{Z}$ and

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Example

For every DG algebra A with H^*A concentrated in degrees $d\mathbb{Z}$, $(H^*A, \{m_{d+2}\})$ is a d-sparse Massey algebra.

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The ***Hochschild–Massey complex*** of a d-sparse Massey algebra (H, m) is

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The d-sparse generalized Kadeishvili theorem

Theorem (Jasso and Muro 2023)

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as graded algebras concentrated in degrees $d\mathbb{Z}$ such that, under this identification,

$$\{m_{d+2}^A\} = \{m_{d+2}^B\}.$$

If

$$\mathrm{HH}^{n+2,-n}(H^*A, \{m_{d+2}^A\}) = 0, \qquad n > d,$$

then

$$A \simeq B.$$

The derived Auslander–Iyama correspondence

Theorem (Derived Auslander–Iyama correspondence, Jasso and Muro 2023)

Suppose that k is perfect and $d \geq 1$. There is a bijection

$$\left\{ \begin{array}{c} \text{DG algebras } A \text{ up to quasi-isomorphism} \\ H^0A \text{ basic finite-dimensional} \\ A \in D^c(A) \text{ a } d\mathbb{Z}\textbf{-cluster tilting object} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} (\Lambda, I) \text{ up to isomorphism} \\ \Lambda \text{ a basic Frobenius algebra} \\ I \text{ an invertible } \Lambda\text{-bimodule} \\ \text{such that } \Omega^{d+2}_{\Lambda^e}(\Lambda) \cong I \text{ stably} \end{array} \right\},$$

$$A \longmapsto (H^0A, H^{-d}A).$$

In this case

$$H^*A \cong \bigoplus_{di \in d\mathbb{Z}} {}_1\Lambda_{\sigma^{-i}} = \bigoplus_{di \in d\mathbb{Z}} {}_1\Lambda_{\sigma}^{\otimes^{-i}} = \frac{\Lambda\langle t^{\pm 1}\rangle}{(t\lambda - \sigma(\lambda)t)_{\lambda \in \Lambda}} = \Lambda(\sigma, d), \quad |t| = -d.$$

Calabi–Yau structures

Let $DA = \mathrm{Hom}_k(A, k)$. A DG algebra A is ***bimodule right n-Calabi–Yau*** if

$$A[n] \cong DA \qquad \text{in } D(A^e).$$

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The converse is a lifting problem:

When does an isomorphism $H^*A[n] \cong DH^*A$ lift to $A[n] \cong DA$?

The Batalin–Vilkovisky obstruction

An isomorphism

$$H^*A[n] \cong DH^*A$$

defines a ***Batalin–Vilkovisky (BV) operator***

$$\Delta : \ \mathrm{HH}^{p,q}(H^*A) \longrightarrow \mathrm{HH}^{p-1,q}(H^*A).$$

The Batalin–Vilkovisky obstruction

An isomorphism

$$H^*A[n] \cong DH^*A$$

defines a ***Batalin–Vilkovisky (BV) operator***

$$\Delta\colon\ \mathrm{HH}^{p,q}(H^*A) \longrightarrow \mathrm{HH}^{p-1,q}(H^*A).$$

Theorem (Jasso and Muro 2026)

Let A be a DG algebra such that H^0A is basic finite-dimensional and $A \in D^c(A)$ is $d\mathbb{Z}$-cluster tilting for some $d \geq 1$. An isomorphism

$$H^*A[sd] \cong DH^*A$$

lifts to a bimodule right sd-Calabi–Yau structure on A if and only if

$$\Delta(\{m_{d+2}\}) = 0 \in \mathrm{HH}^{d+1,-d}(H^*A).$$

Calabi–Yau structures on $\Lambda(\sigma, d)$

For the graded algebra $\Lambda(\sigma, d)$, $d \geq 1$, an isomorphism

$$\Lambda(\sigma, d)[sd] \cong D\Lambda(\sigma, d)$$

is the same as an associative nondegenerate bilinear pairing

$$\langle -, - \rangle \colon \Lambda \times \Lambda \longrightarrow k$$

such that

$$\begin{aligned} \langle y, x \rangle &= \langle \sigma^m(x), y \rangle, \\ \langle \sigma^l(x), y \rangle &= (-1)^{dl(sd+1)} \langle x, \sigma^{-l}(y) \rangle, \quad l \in \mathbb{Z}. \end{aligned}$$

An example with non-vanishing BV obstruction

Let k be a perfect field of char $k = 2$ and let A be the DG algebra corresponding to the algebra

$$\Lambda = \frac{k[\varepsilon]}{(\varepsilon^2)}$$

and the diagonal bimodule Λ under the derived Auslander–Iyama correspondence for some odd $d \geq 1$. In particular $\sigma = \mathrm{id}$.

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Consider the pairing given by

$$\langle 1, 1\rangle = 1, \qquad \langle 1, \varepsilon\rangle = 1, \qquad \langle \varepsilon, 1\rangle = 1, \qquad \langle \varepsilon, \varepsilon\rangle = 0.$$

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This pairing satisfies the previous conditions for $s = 0$ but

$$\Delta(\{m_{d+2}\}) \neq 0.$$

Hence, it does not lift to a bimodule right 0-Calabi–Yau structure on A.

THE END

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