

Triangulated categories with universal Toda bracket

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$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & \text{push} & \downarrow \\ 0 & \longrightarrow & B/A \end{array}$$

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- Which is the right filler?

A Postnikov tower [Dwyer-Kan-Smith]

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$$S = LM \rightleftarrows M$$

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Notice that for any pair of objects A, B ,

$$LM(A, B) \simeq LN(A, B) \quad !!!$$

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A functor $\varphi: \mathbf{D} \rightarrow \mathbf{C}$ induces a homomorphism

$$\varphi^*: H^*(\mathbf{C}, M) \longrightarrow H^*(\mathbf{D}, \varphi^* M),$$

where $\varphi^* M = M(\varphi, \varphi)$.

Cohomology of categories

This can be computed as the cohomology of a cobar-like complex $F^*(\mathbf{C}, M)$ where an n -cochain c is a function sending a chain of n composable morphisms in $\mathbf{C}$

$$A_0 \xleftarrow{\sigma_1} A_1 \leftarrow \cdots \leftarrow A_{n-1} \xleftarrow{\sigma_n} A_n$$

to an element

$$c(\sigma_1, \cdots, \sigma_n) \in M(A_n, A_0).$$

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The **universal Toda bracket** of a stable model category $\mathbf{M}$ is the first k -invariant of LM

$$k_1 \in H^3(\text{Ho } \mathbf{M}, \text{Hom}_{\text{Ho } \mathbf{M}}(\Sigma, -)).$$

Toda brackets

Suppose that $(\mathbf{A}, \Sigma, \mathcal{E})$ is a triangulated category.

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \qquad gf = 0, \quad hg = 0.$$

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Example . $1_{\Sigma A} \in \langle q, i, f \rangle$. *Actually it is immediate to see for $D = \Sigma A$ that the lower triangle is an exact triangle if and only if it is coexact and $1_{\Sigma A} \in \langle h, g, f \rangle$.*

Toda brackets

Example . $-1_{\Sigma B} \in \langle \Sigma f, q, i \rangle,$

$$\begin{array}{ccccccc}
 B & \xrightarrow{i} & C_f & \xrightarrow{q} & \Sigma A & \xrightarrow{-\Sigma f} & \Sigma B \\
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$1_{\Sigma C_f} \in \langle \Sigma i, \Sigma f, q \rangle.$

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$$\mathbf{Toda} = \begin{array}{ccccccc} & & 0 & & & & \\ & \nearrow & & \searrow & & & \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ & & & \nearrow & & \searrow & \\ & & & 0 & & & \end{array} \xrightarrow{\varphi} \mathbf{A}.$$

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$$\mathbf{Toda} = \begin{array}{ccccccc} & & 0 & & & & \\ & \nearrow & & \searrow & & & \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ & & & \nearrow & & \searrow & \\ & & & 0 & & & \end{array} \xrightarrow{\varphi} \mathbf{A}.$$

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$$\begin{array}{ccc} \varphi^* k_1 & \in & H^3(\mathbf{Toda}, \varphi^* \text{Hom}_{\mathbf{A}}(\Sigma, -)) \\ \text{[Baues-Dreckmann]} \parallel & & \parallel \\ \langle h, g, f \rangle & \in & \frac{\mathbf{A}(\Sigma A, D)}{h\mathbf{A}(\Sigma A, C) + \mathbf{A}(\Sigma B, D)(\Sigma f)} \end{array}$$

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Example . Let $\text{free}(S) \subset L\mathbf{Spectra}$ be the full S -category of the simplicial localization of spectra given by

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where S is the sphere spectrum. All triple Toda brackets vanish in $\mathbf{free}(S)$. However, the universal Toda bracket of $\mathbf{free}(S)$ is the generator of

$$H^3(\mathbf{free}(\mathbb{Z}), \mathrm{Hom}(-, - \otimes \mathbb{Z}/2)) \cong H_{ML}^3(\mathbb{Z}, \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

Detecting the exact triangles

Let $\mathbf{A}$ be any additive category, $\Sigma: \mathbf{A} \xrightarrow{\sim} \mathbf{A}$ a self-equivalence, and $\theta \in H^3(\mathbf{A}, \text{Hom}_{\mathbf{A}}(\Sigma, -))$ any cohomology class (which needs not be the universal Toda bracket of any stable model category).

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Let $\mathbb{I} = (0 \rightarrow 1)$ and let $[\mathbb{I}, \mathbf{A}]$ be the category of functors, called **pairs**. Objects are regarded as cochain complexes $d_A: A_0 \rightarrow A_1$ concentrated in dimensions 0 and 1.

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 & & \downarrow \text{K\"unneth SS} \\
 & & \bar{k}_1 \in H^2([\mathbb{I}, \mathbf{A}], H^1 \text{Hom}_{\mathbf{A}}(\Sigma, -))
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The image $\bar{k}_1$ of k_1 determines a **linear extension** called the category of **homotopy pairs**,

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In particular given a morphism $f: A \rightarrow B$ and an object X in $\mathbf{A}$ there is a long exact sequence

$$(S) \quad \mathbf{A}(\Sigma B, U) \xrightarrow{(\Sigma f)^*} \mathbf{A}(\Sigma A, X) \rightarrow [\mathbb{I}, \mathbf{B}](f, 0 \rightarrow X) \rightarrow \mathbf{A}(B, X) \xrightarrow{f^*} \mathbf{A}(A, X).$$

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Theorem . *For $\mathbf{A} = \mathbf{Ho M}$ the triangles (T) are the exact triangles.*

Detecting the exact triangles

- Are there conditions on θ which imply that the triangles (T) induce a triangulated structure in $\mathbf{A}$ with translation functor Σ ?

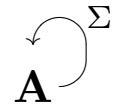
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The situation when the triangles (T) define a triangulated structure is very convenient since, for example, cofibers are automatically functorial in the category $[\mathbb{I}, \mathbf{B}]$. One can also construct the differential d_2 of Adams spectral sequence **[Baues-Jibladze]**. . .

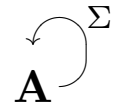
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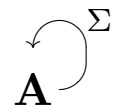


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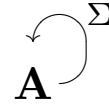
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The **diagram cohomology** of Σ with coefficients in $\bar{\Sigma}$ can be obtained as

$$H^*(\Sigma, \bar{\Sigma}) = H^* \operatorname{Fib} \left(\Sigma^* - \bar{\Sigma}_*: F^*(\mathbf{A}, \operatorname{Hom}_{\mathbf{A}}(\Sigma, -)) \longrightarrow F^*(\mathbf{A}, \operatorname{Hom}_{\mathbf{A}}(\Sigma^2, \Sigma)) \right).$$

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In particular there is a long exact sequence

$$\cdots \rightarrow H^n(\Sigma, \bar{\Sigma}) \xrightarrow{j} H^n(\mathbf{A}, \operatorname{Hom}_{\mathbf{A}}(\Sigma, -)) \xrightarrow{\Sigma^* - \bar{\Sigma}_*} H^n(\mathbf{A}, \operatorname{Hom}_{\mathbf{A}}(\Sigma^2, \Sigma)) \rightarrow H^{n+1}(\Sigma, \bar{\Sigma}) \rightarrow \cdots$$

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This k -invariant for diagrams is completely determined by $k_2 \in H^4(P_1 L\mathbf{M}, \pi_2 L\mathbf{M})$.

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Theorem . [Baues-M.] *In the conditions above if $\nabla \in D^{1,2}$ then $\mathbf{A}$ with the triangles (T) above satisfies all axioms except from the octahedral axiom. Moreover, if $\nabla \in D^{2,1}$ then the octahedral axiom is also satisfied and hence the triangles (T) yield a triangulated structure on $\mathbf{A}$.*

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Definition . A *cohomologically triangulated category* is a triple $(\mathbf{A}, \Sigma, \nabla)$ where $\mathbf{A}$ is an additive category, $\Sigma: \mathbf{A} \xrightarrow{\sim} \mathbf{A}$ is a self-equivalence, and $\nabla \in H^3(\Sigma, \bar{\Sigma})$ satisfying the second condition in the Theorem, so that the universal Toda bracket $j\nabla \in H^3(\mathbf{A}, \text{Hom}_{\mathbf{A}}(\Sigma, -))$ induces a triangulated structure in $\mathbf{A}$.

The last example

Consider $\mathbf{M} = \mathbf{mod}(\mathbb{Z}/p^2)$ and $\mathbf{N} = \mathbf{mod}(\mathbb{Z}/p[t]/t^2)$. Recall that in both cases the homotopy category is $\mathbf{A} = \mathbf{mod}(\mathbb{Z}/p)$, the suspension functor is the identity $\Sigma = 1$, and $k_1 = 0$.

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$$H^2(\mathbf{mod}(\mathbb{Z}/p), \mathrm{Hom}) \xrightarrow{\Sigma^* - \bar{\Sigma}_*} H^2(\mathbf{mod}(\mathbb{Z}/p), \mathrm{Hom}) \longrightarrow H^3(\Sigma, \bar{\Sigma}) \longrightarrow 0$$

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 \cong \uparrow & & \cong \uparrow & & \\
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 \end{array}$$

Therefore

$$H^3(\Sigma, \bar{\Sigma}) = \begin{cases} \mathbb{Z}/2, & p = 2, \\ 0, & p \neq 2. \end{cases}$$

The last example

For $p = 2$ one can check that $\nabla^{\mathbf{N}} = 0$ and $\nabla^{\mathbf{M}} \neq 0$, hence the cohomologically triangulated structures associated to $\mathbf{M} = \mathbf{mod}(\mathbb{Z}/p^2)$ and $\mathbf{N} = \mathbf{mod}(\mathbb{Z}/p[t]/t^2)$ are different

$(\mathbf{mod}(\mathbb{Z}/2), \Sigma, 1), (\mathbf{mod}(\mathbb{Z}/2), \Sigma, 0)$, respectively,

and $k_2^{\mathbf{M}} \neq k_2^{\mathbf{N}}$.

The End

Thanks for your attention!