Triangulated categories with universal Toda bracket

Fernando Muro

(joint with H.-J. Baues)

Max-Planck-Institut für Mathematik, Bonn

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$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\text{push}"] & B \arrow[d, two heads] \\ 0 \arrow[r] & B/A \end{tikzcd}$$

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- Which is the right filler?

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Notice that for any pair of objects A, B,

$$L\mathbf{M}(A, B) \simeq L\mathbf{N}(A, B) \quad !!!$$

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A functor $\varphi\colon \mathbf{D} \to \mathbf{C}$ induces a homomorphism

$$\varphi^*\colon H^*(\mathbf{C}, M) \longrightarrow H^*(\mathbf{D}, \varphi^* M),$$

where $\varphi^* M = M(\varphi, \varphi)$.

Cohomology of categories

This can be computed as the cohomology of a cobar-like complex $F^*(\mathbf{C}, M)$ where an n-cochain c is a function sending a chain of n composable morphisms in $\mathbf{C}$

$$A_0 \xleftarrow{\sigma_1} A_1 \leftarrow \cdots \leftarrow A_{n-1} \xleftarrow{\sigma_n} A_n$$

to an element

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The universal Toda bracket of a stable model category $\mathbf{M}$ is the first k-invariant of $L\mathbf{M}$

$$k_1 \in H^3(\mathrm{Ho}\,\mathbf{M}, \mathrm{Hom}_{\mathrm{Ho}\,\mathbf{M}}(\Sigma, -)).$$

Toda brackets

Suppose that $(\mathbf{A}, \Sigma, \mathcal{E})$ is a triangulated category.

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \qquad\qquad gf = 0,\ hg = 0.$$

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Example . $1_{\Sigma A} \in \langle q, i, f \rangle$. *Actually it is immediate to see for* $D = \Sigma A$ *that the lower triangle is an exact triangle if and only if it is coexact and* $1_{\Sigma A} \in \langle h, g, f \rangle$.

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Example . $-1_{\Sigma B} \in \langle \Sigma f, q, i \rangle$,

$$\begin{array}{ccccccc}
B & \xrightarrow{i} & C_f & \xrightarrow{q} & \Sigma A & \xrightarrow{-\Sigma f} & \Sigma B \\
\| & & \| & & \| & & \downarrow{\scriptstyle -1} \\
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Example . $-1_{\Sigma B} \in \langle \Sigma f, q, i \rangle,$

$$\begin{array}{ccccccc}
B & \xrightarrow{i} & C_f & \xrightarrow{q} & \Sigma A & \xrightarrow{-\Sigma f} & \Sigma B \\
\| & & \| & & \| & & \downarrow {\scriptstyle -1} \\
B & \xrightarrow{i} & C_f & \xrightarrow{q} & \Sigma A & \xrightarrow{\Sigma f} & \Sigma B
\end{array}$$

$1_{\Sigma C_f} \in \langle \Sigma i, \Sigma f, q \rangle.$

Toda brackets

Suppose that our triangulated category is $\mathbf{A} = \mathrm{Ho}\,\mathbf{M}$. Then $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ with $gf = 0$, $hg = 0$, is the same as a functor

$$\mathbf{Toda} \;=\; \begin{tikzcd} \bullet \arrow[r] \arrow[rr, bend left, "0"] & \bullet \arrow[r] \arrow[rr, bend right, "0"'] & \bullet \arrow[r] & \bullet \end{tikzcd} \xrightarrow{\varphi} \mathbf{A}.$$

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$$\begin{array}{ccc} \varphi^* k_1 & \in & H^3(\mathbf{Toda}, \varphi^* \operatorname{Hom}_{\mathbf{A}}(\Sigma, -)) \\ \textbf{[Baues-Dreckmann]}\;\Big\| & & \Big\| \\ \langle h, g, f \rangle & \in & \dfrac{\mathbf{A}(\Sigma A, D)}{h\mathbf{A}(\Sigma A, C) + \mathbf{A}(\Sigma B, D)(\Sigma f)} \end{array}$$

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Example . *Let* $\mathbf{free}(S) \subset L\mathbf{Spectra}$ *be the full* S*-category of the simplicial localization of spectra given by*

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$$H^3(\mathbf{free}(\mathbb{Z}), \mathrm{Hom}(-, - \otimes \mathbb{Z}/2)) \cong H^3_{ML}(\mathbb{Z}, \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

Detecting the exact triangles

Let $\mathbf{A}$ be any additive category, $\Sigma\colon \mathbf{A} \xrightarrow{\sim} \mathbf{A}$ a self-equivalence, and $\theta \in H^3(\mathbf{A}, \mathrm{Hom}_{\mathbf{A}}(\Sigma, -))$ any cohomology class (which needs not be the universal Toda bracket of any stable model category).

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The image $\bar{k}_1$ of k_1 determines a linear extension called the category of homotopy pairs,

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In particular given a morphism $f\colon A \to B$ and an object X in $\mathbf{A}$ there is a long exact sequence

(S) $$\mathbf{A}(\Sigma B, U) \stackrel{(\Sigma f)^*}{\to} \mathbf{A}(\Sigma A, X) \to [\mathbb{I}, \mathbf{B}](f, 0 \to X) \to \mathbf{A}(B, X) \stackrel{f^*}{\to} \mathbf{A}(A, X).$$

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Suppose that $[\mathbb{I}, \mathbf{B}](f, 0 \to X)$ is representable as a functor in X,

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Theorem . *For $\mathbf{A} = \mathrm{Ho}\,\mathbf{M}$ the triangles (T) are the exact triangles.*

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- Are there conditions on θ which imply that the triangles (T) induce a triangulated structure in $\mathbf{A}$ with translation functor Σ?

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The situation when the triangles (T) define a triangulated structure is very convenient since, for example, cofibers are automatically functorial in the category $[\mathbb{I}, \mathbf{B}]$. One can also construct the differential d_2 of Adams spectral sequence **[Baues-Jibladze]**. . .

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The diagram cohomology of Σ with coefficients in $\bar{\Sigma}$ can be obtained as

$$H^*(\Sigma, \bar{\Sigma}) \quad = \quad H^*\mathsf{Fib}\left(\Sigma^* - \bar{\Sigma}_*\colon F^*(\mathbf{A}, \operatorname{Hom}_{\mathbf{A}}(\Sigma, -)) \longrightarrow F^*(\mathbf{A}, \operatorname{Hom}_{\mathbf{A}}(\Sigma^2, \Sigma))\right).$$

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In particular there is a long exact sequence

$$\cdots \to H^n(\Sigma, \bar{\Sigma}) \xrightarrow{j} H^n(\mathbf{A}, \mathrm{Hom}_{\mathbf{A}}(\Sigma, -)) \xrightarrow{\Sigma^* - \bar{\Sigma}_*} H^n(\mathbf{A}, \mathrm{Hom}_{\mathbf{A}}(\Sigma^2, \Sigma)) \to H^{n+1}(\Sigma, \bar{\Sigma}) \to \cdots$$

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This k-invariant for diagrams is completely determined by $k_2 \in H^4(P_1 L\mathbf{M}, \pi_2 L\mathbf{M})$.

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Theorem . [Baues-M.] *In the conditions above if $\nabla \in D^{1,2}$ then $\mathbf{A}$ with the triangles (T) above satisfies all axioms except from the octahedral axiom. Moreover, if $\nabla \in D^{2,1}$ then the octahedral axiom is also satisfied and hence the triangles (T) yield a triangulated structure on $\mathbf{A}$.*

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Definition . *A cohomologically triangulted category is a triple $(\mathbf{A}, \Sigma, \nabla)$ where $\mathbf{A}$ is an additive category, $\Sigma\colon \mathbf{A} \xrightarrow{\sim} \mathbf{A}$ is a self-equivalence, and $\nabla \in H^3(\Sigma, \bar{\Sigma})$ satisfying the second condition in the Theorem, so that the universal Toda bracket $j\nabla \in H^3(\mathbf{A}, \mathrm{Hom}_{\mathbf{A}}(\Sigma, -))$ induces a triangulated structure in $\mathbf{A}$.*

The last example

Consider $\mathbf{M} = \mathbf{mod}(\mathbb{Z}/p^2)$ and $\mathbf{N} = \mathbf{mod}(\mathbb{Z}/p[t]/t^2)$. Recall that in both cases the homotopy category is $\mathbf{A} = \mathbf{mod}(\mathbb{Z}/p)$, the suspension functor is the identity $\Sigma = 1$, and $k_1 = 0$.

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$$H^2(\mathbf{mod}(\mathbb{Z}/p), \mathrm{Hom}) \xrightarrow{\Sigma^* - \bar{\Sigma}_*} H^2(\mathbf{mod}(\mathbb{Z}/p), \mathrm{Hom}) \longrightarrow H^3(\Sigma, \bar{\Sigma}) \longrightarrow 0$$

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\cong \big\uparrow & & \cong \big\uparrow & & & & \\
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\cong \uparrow & & \cong \uparrow & & & & \\
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\cong \uparrow & & \cong \uparrow & & & & \\
\mathbb{Z}/p & \xrightarrow{2} & \mathbb{Z}/p & & & &
\end{array}$$

Therefore

$$H^3(\Sigma, \bar{\Sigma}) = \begin{cases} \mathbb{Z}/2, & p = 2, \\ 0, & p \neq 2. \end{cases}$$

The last example

For $p = 2$ one can check that $\nabla^{\mathbf{N}} = 0$ and $\nabla^{\mathbf{M}} \neq 0$, hence the cohomologically triangulated structures associated to $\mathbf{M} = \mathbf{mod}(\mathbb{Z}/p^2)$ and $\mathbf{N} = \mathbf{mod}(\mathbb{Z}/p[t]/t^2)$ are different

$$(\mathbf{mod}(\mathbb{Z}/2), \Sigma, 1), \quad (\mathbf{mod}(\mathbb{Z}/2), \Sigma, 0), \text{ respectively,}$$

and $k_2^{\mathbf{M}} \neq k_2^{\mathbf{N}}$.

The End

Thanks for your attention!