Enhanced n-angulated categories

Fernando Muro *joint work in progress with* Gustavo Jasso

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Universidad de Sevilla

Defined by Geiss, Keller, and Oppermann, 2013.

A 3-angulated category is a triangulated category.

No higher categories.

Just longer 'triangles' called n-angles

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} \Sigma X_n .$$

An *n*-angulated category is an additive category $\mathcal{C}$ equipped with a self-equivalence

$$\Sigma \colon \mathcal{C} \xrightarrow{\sim} \mathcal{C},$$

called suspension, and a class of diagrams

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} \Sigma X_n,$$

called exact *n*-angles, satisfying the following axioms.

n-angulated categories

Exact n-angles are closed under direct sums, direct summands, and isomorphisms.

The following trivial n-angle is exact

$$A \xrightarrow{\mathrm{id}_A} A \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Sigma A.$$

Any morphism $f_n \colon X_n \to X_{n-1}$ is the base of an exact n-angle

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} \Sigma X_n.$$

An n-angle is exact if and only if its rotation is exact,

$$X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} \Sigma X_n \xrightarrow{(-1)^n \Sigma f_n} \Sigma X_{n-1}.$$

n-angulated categories

Any commutative square between the bases of two exact n-angles extends to a morphism of n-angles

$$\begin{array}{ccccccccccccc}
X_n & \xrightarrow{f_n} & X_{n-1} & \xrightarrow{f_{n-1}} & X_{n-2} & \xrightarrow{f_{n-2}} & \cdots & \xrightarrow{f_2} & X_1 & \xrightarrow{f_1} & \Sigma X_n \\
\downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_{n-2}} & & & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \Sigma\varphi_n} \\
Y_n & \xrightarrow{g_n} & Y_{n-1} & \xrightarrow{g_{n-1}} & Y_{n-2} & \xrightarrow{g_{n-2}} & \cdots & \xrightarrow{g_2} & Y_1 & \xrightarrow{g_1} & \Sigma Y_n
\end{array}$$

This can be done in such a way that the mapping cone is exact

$$X_{n-1} \oplus Y_n \xrightarrow{\begin{pmatrix} -f_{n-1} & 0 \\ \varphi_{n-1} & g_n \end{pmatrix}} X_{n-2} \oplus Y_{n-1} \xrightarrow{\begin{pmatrix} -f_{n-2} & 0 \\ \varphi_{n-2} & g_{n-1} \end{pmatrix}} \cdots$$

$$\cdots \xrightarrow{\begin{pmatrix} -f_1 & 0 \\ \varphi_1 & g_2 \end{pmatrix}} \Sigma X_n \oplus Y_1 \xrightarrow{\begin{pmatrix} -\Sigma f_n & 0 \\ \Sigma\varphi_n & g_1 \end{pmatrix}} \Sigma X_{n-1} \oplus \Sigma Y_n.$$

n-angulated subcategories of triangulated categories

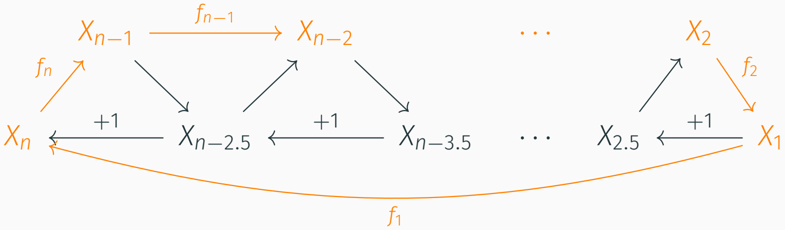
Let $\mathcal{T}$ be a small idempotent-complete triangulated category with suspension Σ and $\mathcal{C} \subset \mathcal{T}$ a full subcategory closed under direct sums and summands satisfying:

- $\Sigma^{n-2}\mathcal{C} = \mathcal{C}$.
- $\mathcal{T}(X, \Sigma^i Y) = 0$ for $X, Y \in \mathcal{C}$ and $(n-2) \nmid i$.
- $\mathcal{T} = \langle \mathcal{C} \rangle$.

The condition on idempotents is not strong by Balmer and Schlichting, 2001; Lin, 2021.

n-angulated subcategories of triangulated categories

We equip $\mathcal{C}$ with the suspension Σ^{n-2} and consider the n-angles in $\mathcal{C}$ fitting in a diagram in $\mathcal{T}$ with exact and commutative triangles



We say that $\mathcal{C} \subset \mathcal{T}$ is a generating n-angulated subcategory if it is an n-angulated category with these exact n-angles.

Theorem [Geiss, Keller, and Oppermann, 2013]

Let $\mathcal{T}$ be a triangulated category and $\mathcal{C} \subset \mathcal{T}$ an $(n-2)$-cluster tilting subcategory, in the sense of Iyama and Yoshino, 2008, satisfying $\Sigma^{n-2}\mathcal{C} = \mathcal{C}$. Then $\mathcal{C} \subset \mathcal{T}$ is an n-angulated subcategory.

Any object $X \in \mathcal{T}$ can be inductively constructed from $\mathcal{C}$ in $n-2$ steps by means of exact triangles

$$\Sigma^{i-1}C_i \longrightarrow X_{i-1} \longrightarrow X_i \longrightarrow \Sigma^i C_i, \qquad 0 \leq i \leq n-3,$$

with $X_{-1} = 0$, $X_{n-3} = X$, and $C_i \in \mathcal{C}$.

An enhanced n-angulated category $\mathcal{A}$ is a DG-category such that the Yoneda inclusion

$$\begin{aligned} H^0(\mathcal{A}) &\longrightarrow D^c(\mathcal{A}), \\ X &\mapsto \mathcal{A}(-,X), \end{aligned}$$

is the inclusion of an n-angulated subcategory.

This extends Bondal and Kapranov, 1991 for $n = 3$.

An n-angulated category $\mathcal{C}$ is algebraic if $\mathcal{C} \simeq H^0(\mathcal{A})$ for some enhanced n-angulated category $\mathcal{A}$.

This extends Keller, 2007 for $n = 3$. Compare Jasso, 2016.

Proposition

If $\mathcal{T}$ is an algebraic triangulated category and $\mathcal{C} \subset \mathcal{T}$ is an n-angulated subcategory then $\mathcal{C}$ is also algebraic.

There are non-algebraic examples in Bergh, Jasso, and Thaule, 2016 based in Muro, Schwede, and Strickland, 2007.

Let Λ be a finite-dimensional basic self-injective algebra and

$$_{\sigma^{-1}}\Lambda_1 \hookrightarrow P_n \to \cdots \to P_1 \twoheadrightarrow \Lambda$$

an extension of Λ-bimodules with $\sigma\colon \Lambda \cong \Lambda$ an automorphism and projective-injective middle terms, i.e. $\Omega^n_{\Lambda^{\mathrm{env}}}\Lambda \cong {}_{\sigma^{-1}}\Lambda_1$ stably.

The functor

$$- \otimes_\Lambda {}_\sigma\Lambda_1 \colon \operatorname{proj}(\Lambda) \xrightarrow{\sim} \operatorname{proj}(\Lambda)$$

is an equivalence with inverse $- \otimes_\Lambda {}_{\sigma^{-1}}\Lambda_1$.

Theorem [Lin, 2019]

In the previous setting, the category proj(Λ) equipped with the suspension functor $- \otimes_{\Lambda} {}_{\sigma}\Lambda_1$ and the exact n-angles described below is n-angulated.

This extends Amiot, 2007 for $n = 3$.

If proj(Λ) is n-angulated then Λ is self-injective by Geiss, Keller, and Oppermann, 2013.

n-angulated categories and self-injective algebras

An n-angle in proj(Λ) is exact if the extended sequence is exact

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_n \otimes_\Lambda {}_\sigma\Lambda_1 \xrightarrow{f_n \otimes_\Lambda {}_\sigma\Lambda_1} X_{n-1} \otimes_\Lambda {}_\sigma\Lambda_1$$

and the induced extension

$$M \otimes_\Lambda {}_{\sigma^{-1}}\Lambda_1 \hookrightarrow X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \twoheadrightarrow M,$$

with $M = \operatorname{coker} f_2 = \ker f_n \otimes_\Lambda {}_\sigma\Lambda_1$, is equivalent to

$$M \otimes_\Lambda ({}_{\sigma^{-1}}\Lambda_1 \hookrightarrow P_n \to \cdots \to P_1 \twoheadrightarrow \Lambda).$$

Theorem

Let Λ be a finite-dimensional basic self-injective algebra over a perfect field.

1. $\operatorname{proj}(\Lambda)$ is an algebraic n-angulated category if and only if $\Omega^n_{\Lambda^{\mathrm{env}}}\Lambda \cong {}_{\sigma^{-1}}\Lambda_1$ stably for some automorphism σ.
2. The possible suspension functors are $- \otimes_\Lambda {}_\sigma\Lambda_1$.
3. If we fix the suspension functor, there is a unique n-angulated enhancement up to quasi-equivalence.

Corollary

Let $\mathcal{T}$ be an idempotent-complete algebraic triangulated category over a perfect field with a basic $(n-2)$-cluster tilting object C satisfying $\Sigma^{n-2}C = C$. The associated $(n-2)$-cluster tilting subcategory $\mathcal{C} \subset \mathcal{T}$, which is n-angulated by Geiss, Keller, and Oppermann, 2013, is algebraic and has an essentially unique enhancement.

In this case $\mathcal{C} = \mathrm{proj}(\mathcal{T}(C, C))$.

Corollary

Let $\mathcal{T}$ be an idempotent-complete algebraic triangulated category over a perfect field with a basic $(n-2)$-cluster tilting object C satisfying $\Sigma^{n-2}C = C$. Then $\mathcal{T}$ has a unique enhancement up to Morita equivalence.

Proof.

Let $\mathcal{C} \subset \mathcal{T}$ be the completion of C by direct sums and direct summands.

If $\mathcal{A}$ is a 3-angulated enhancement of $\mathcal{T} = H^0(\mathcal{A})$ then the full sub-DG-category $\mathcal{B} \subset \mathcal{A}$ spanned by the objects of $\mathcal{C}$ is an n-angulated enhancement of $\mathcal{C}$.

Since $\mathcal{T} = \langle \mathcal{C} \rangle$, $\mathcal{A}$ is the enhanced triangulated envelope of $\mathcal{B}$ in the sense of Bondal and Kapranov, 1991. Hence, the uniqueness of $\mathcal{B}$ implies the uniqueness of $\mathcal{A}$. □

Applications

If $\mathcal{T}$ is a triangulated category and $\mathcal{C} \subset \mathcal{T}$ is an n-angulated subcategory, we say that $\mathcal{T}$ is a triangulated envelope of the n-angulated category $\mathcal{C}$.

Corollary

Let Λ be a finite-dimensional basic self-injective algebra over a perfect field. If $\mathcal{C} = \mathrm{proj}(\Lambda)$ is n-angulated then it has an essentially unique algebraic triangulated envelope $\mathcal{T}$ and $\mathcal{C} \subset \mathcal{T}$ is $(n-2)$-cluster tilting.

How to prove the main theorem

Let Λ be a finite-dimensional basic self-injective algebra and

$$ {}_{\sigma^{-1}}\Lambda_1 \hookrightarrow P_n \to \cdots \to P_1 \twoheadrightarrow \Lambda $$

an extension of Λ-bimodules with $\sigma\colon \Lambda \cong \Lambda$ an automorphism and projective-injective middle terms, i.e. $\Omega^n_{\Lambda^{\mathrm{env}}}\Lambda \cong {}_{\sigma^{-1}}\Lambda_1$ stably.

If $\mathcal{A}$ is an enhancement of $\operatorname{proj}(\Lambda) = H^0(\mathcal{A})$ then

$$ H^*\mathcal{A}(\Lambda,\Lambda) = \Lambda(\sigma) := \frac{\Lambda\langle t^{\pm 1}\rangle}{(t\lambda - \sigma(\lambda)t)}, \qquad |t| = 2-n. $$

How to prove the main theorem

Quasi-equivalence classes of enhancements of proj(Λ) are in bijection with gauge equivalence classes of certain minimal A_∞-algebra structures on $\Lambda(\sigma)$, given by degree $2-i$ operations

$$m_i \colon \Lambda(\sigma)\otimes \overset{i}{\cdots} \otimes\Lambda(\sigma) \longrightarrow \Lambda(\sigma), \qquad i \geq 3,$$

satisfying certain equations.

How to prove the main theorem

The first possibly non-trivial operation defines a Hochschild cohomology class

$$\{m_n\} \in HH^{n,2-n}(\Lambda(\sigma), \Lambda(\sigma)).$$

The restriction along the inclusion $\Lambda \subset \Lambda(\sigma)$ of the degree 0 part

$$\{m_n\}_{|\Lambda} \in HH^n(\Lambda, {}_{\sigma^{-1}}\Lambda_1) = \operatorname{Ext}^n_{\Lambda^{\mathrm{env}}}(\Lambda, {}_{\sigma^{-1}}\Lambda_1)$$

must be a representative of the previous bimodule extension.

How to prove the main theorem

We show that there exists a unique class

$$x \in HH^{n,2-n}(\Lambda(\sigma), \Lambda(\sigma))$$

restricting to the given extension in $\mathrm{Ext}^n_{\Lambda^{\mathrm{env}}}(\Lambda, {}_{\sigma^{-1}}\Lambda_1)$ and satisfying

$$\frac{[x,x]}{2} = 0 \in HH^{2n-1,2(2-n)}(\Lambda(\sigma), \Lambda(\sigma)).$$

This is the first obstruction to the extension of a cocycle m_n representing x to an A_∞-algebra structure.

How to prove the main theorem

Higher obstructions live in the subsequent pages a spectral sequence with

$$E_2^{pq} = HH^{p+2,q}(\Lambda(\sigma), \Lambda(\sigma)), \qquad p > 0,$$

a posteriori converging to the homotopy groups of the moduli space of enhancements.

The given extension is a unit in Hochschild–Tate cohomology

$$\widehat{HH}^{*,*}(\Lambda, \Lambda(\sigma)).$$

This is used to prove that the spectral sequence collapses in the third page and all remaining obstructions vanish.

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Thanks for your attention!

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