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On the classification of triangulated categories with finiteness conditions

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1. Additively finite triangulated categoires

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Definitions

additively finite category $\quad \mathcal{C} \xrightarrow{\sim} \operatorname{proj}\Lambda \quad$ f.d. projective (right) Λ-modules

basic additive generator $\quad c \longmapsto \Lambda \cong \mathcal{C}(c,c) \quad$ Lambda algebra (f.d. basic)

differential graded algebra $\quad A \rightsquigarrow D(A) \quad$ derived category of A-modules

(small) algebraic $\quad \mathcal{T} \xrightarrow{\sim} \operatorname{per} A \subset D(A) \quad$ thick subcategory generated by A

triangulated category $\quad c \longmapsto A \quad$ enhancement

quasi-isomorphism $\quad A \simeq B \Rightarrow \operatorname{per} A \simeq \operatorname{per} B \quad$ triangulated equivalence

 $\not\Leftarrow \quad A \mapsto B$

Classify additively finite algebraic triangulated categories and their enhancements

Examples

1. $\underline{\mathrm{mod}}\,\Gamma$ stable module category

   Γ f.d. self-injective algebra of finite representation type

$\Lambda = \underline{\mathrm{End}}(M)$ stable Auslander algebra of Γ, $M = M_1 \oplus \cdots \oplus M_n$ indecomposable non-projectives

$A = \mathcal{E}\mathrm{nd}(P)$ P complete resolution of M

2. $\underline{\mathrm{CM}}\,R$ stable category of maximal Cohen-Macaulay (CM) modules

   R CM algebra of finite CM representation type

3. $\mathrm{proj}\,\Lambda$ deformed preprojective algebra of generalized Dynkin type ADEL

   A ? not born algebraic

Λ must be self-injective [Freyd'66]

Auslander-Reiten quiver of $\mathcal{T}$ [Xiao-Zhu'04] $k=\bar{k}$

Uniqueness of enhancements in the standard algebraic case [Amiot'07, Keller'18] $k=\bar{k}$

Theorem (triangulated Auslander correspondence) [Hanihara'20] k perfect TFAE

[Amiot'07]

1. $\operatorname{proj}\Lambda$ admits a triangulated category structure
2. Λ is twisted 3-periodic: in $\underline{\operatorname{mod}}\,\Lambda^e$ $\Omega^3_{\Lambda^e}\Lambda \cong {}_1\Lambda_\sigma$ twisted Λ-bimodule $\sigma \in \operatorname{Aut}(\Lambda)$

Theorem [M'22] Λ connected non-separable Λ/J_Λ separable TFAE k arbitrary

1. Λ is twisted 3-periodic
2. $\exists!$ dga A such that $\operatorname{proj}\Lambda \simeq \operatorname{per} A : \Lambda \mapsto A$

Hochschild cohomology

Amiot triangulated structure induced by

$$\boxed{{}_1\Lambda_\sigma \overset{i}{\hookrightarrow} \underbrace{P_3 \to P_2 \to P_1}_{\in \operatorname{proj}\Lambda^e} \overset{p}{\twoheadrightarrow} \Lambda} \in \operatorname{Ext}^3_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma) = HH^3(\Lambda, {}_1\Lambda_\sigma)$$

- shift $X[-1] = X \otimes_\Lambda {}_1\Lambda_\sigma$
- exact triangles $M \otimes_\Lambda (P_1 \otimes_\Lambda {}_1\Lambda_\sigma \to P_3 \to P_2 \to P_1)$ $M \in \operatorname{mod}\Lambda$

$P_1 \otimes_\Lambda {}_1\Lambda_\sigma \twoheadrightarrow {}_1\Lambda_\sigma \overset{i}{\hookrightarrow} P_3$

meta-triangle

Algebraic ⇒ Amiot

$\mathrm{proj}\,\Lambda \simeq \mathcal{T} \simeq \mathrm{per}\,A : \Lambda \mapsto c \mapsto A$ $\quad X[-1] = X \otimes_\Lambda {}_1\Lambda_\sigma \quad \sigma \in \mathrm{Aut}(\Lambda)$

graded Lambda algebra

$$H^*A = \bigoplus_{n\in\mathbb{Z}} \mathcal{T}(c, c[n]) = \bigoplus_{n\in\mathbb{Z}} {}_1\Lambda_\sigma^{\otimes_\Lambda(-n)} = \bigoplus_{n\in\mathbb{Z}} {}_1\Lambda_{\sigma^{-n}} = \Lambda(\sigma) = \frac{\Lambda\langle t^{\pm 1}\rangle}{(\lambda t - t\sigma(\lambda))_{\lambda\in\Lambda}} \quad |t| = -1$$

minimal model $(\Lambda(\sigma), m_3, m_4, \dots, m_n, \dots)$ A_∞-algebra [Kadeishvili '88]

$m_n : \Lambda(\sigma) \otimes \overset{n}{\cdots} \otimes \Lambda(\sigma) \longrightarrow \Lambda(\sigma)$ $\quad |m_n| = 2-n \quad$ + equations

$j : \Lambda \hookrightarrow \Lambda(\sigma)$ inclusion of degree 0 part [Baues-Dreckmann '89, Benson-Krause-Schwede '04 ...]

$$j^* : HH^{3,-1}(\Lambda(\sigma), \Lambda(\sigma)) \longrightarrow HH^{3,-1}(\Lambda, \Lambda(\sigma)) = HH^3(\Lambda, {}_1\Lambda_\sigma) = \mathrm{Ext}^3_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma)$$

universal Massey product (UMP) $\{m_3\} \longmapsto j^*\{m_3\}$ restricted universal Massey product (rUMP)

The algebraic triangulated structure on $\mathrm{proj}\,\Lambda$ is Amiot's for the rUMP $j^*\{m_3\}$

$j^*\{m_3\}$ represented by ${}_1\Lambda_\sigma \overset{i}{\hookrightarrow} \underbrace{P_3 \to P_2 \to P_1}_{\in\,\mathrm{proj}\,\Lambda^e} \overset{p}{\twoheadrightarrow} \Lambda \Leftrightarrow j^*\{m_3\}$ unit in $\underline{HH}^{\bullet,*}(\Lambda, \Lambda(\sigma))$

Hochschild-Tate cohomology

Example (cusp) $\quad k = \mathbb{C}$

$\mathcal{T} = \underline{\mathrm{CM}}(R) \qquad R = k[[x,y]]/(y^2 - x^3) \qquad c = (x,y)$ additive generator

$\Lambda \cong k[\varepsilon]/(\varepsilon^2) \qquad \sigma(\varepsilon) = -\varepsilon \qquad \Lambda(\sigma) = k\langle \varepsilon, t^{\pm 1}\rangle/(\varepsilon^2, \varepsilon t + t\varepsilon) \qquad |t| = -1$

stable Auslander algebra $\qquad$ graded Lambda algebra

$$A = \mathcal{E}nd\left(\cdots \to R^2 \xrightarrow{\begin{pmatrix} y & x \\ x^2 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & -x \\ -x^2 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & x \\ x^2 & y \end{pmatrix}} R^2 \to \cdots\right)$$ enhancement (2-periodic)

$$\varepsilon: \quad \begin{pmatrix} 1 & y+x^2 \\ y+x^2 & 1 \end{pmatrix} \quad \begin{pmatrix} y & 1-y \\ 1-x^2 & x^2 \end{pmatrix} \quad \begin{pmatrix} 1 & y+x^2 \\ y+x^2 & 1 \end{pmatrix} \quad \begin{pmatrix} y & 1-y \\ 1-x^2 & x^2 \end{pmatrix}$$

$\underline{HH}^{\bullet,*}(\Lambda, \Lambda(\sigma)) = k[u^{\pm 1}, \varepsilon t, t^{\pm 2}] \qquad |u| = (1,-1) \qquad |\varepsilon t| = (0,-1) \qquad |t^2| = (0,-2)$

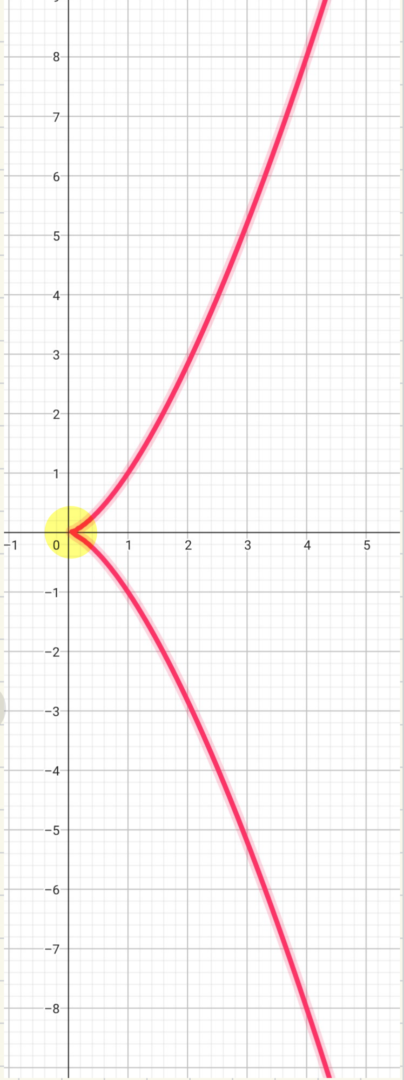
Hochschild-Tate cohomology

rUMP $\quad u^3 t^{-2}$

minimal model $\qquad u: {}_1\Lambda_\sigma \hookrightarrow \Lambda \otimes \Lambda \xrightarrow{\text{mult.}} \Lambda \in \mathrm{Ext}^1_{\Lambda^e}({}_1\Lambda, {}_1\Lambda_\sigma)$

$(\Lambda(\sigma), m_3, 0 \ldots)$ A_∞-algebra $\qquad 1 \mapsto \varepsilon \otimes 1 - 1 \otimes \varepsilon$

$m_3(\varepsilon, \varepsilon, \varepsilon) = t \qquad m_3(\ldots 1 \ldots) = 0 \qquad m_3$ $k\langle t^{\pm 1}\rangle$-linear



Examples (the role of the separability condition) [Jasso-M.'23]

1. $\mathcal{T} = \underline{\mathrm{mod}}\, k[\varepsilon]/(\varepsilon^2)$ $c = k$ additive generator graded Lambda algebra Hochschild-Tate cohomology

$\Lambda = \underline{\mathrm{End}}(k) = k \Rightarrow \mathcal{T} = \mathrm{mod}\, k$ $\sigma = \mathrm{id}$ $\Lambda(\sigma) = k\langle t^{\pm 1}\rangle\ |t| = -1$ $\underline{HH}^{\bullet,*}(\Lambda, \Lambda(\sigma)) = 0$

stable Auslander algebra

$A = \mathcal{E}\mathrm{nd}(\cdots \to k[\varepsilon]/(\varepsilon^2) \xrightarrow{\varepsilon} k[\varepsilon]/(\varepsilon^2) \xrightarrow{\varepsilon} k[\varepsilon]/(\varepsilon^2) \to \cdots) \xleftarrow{\sim} k\langle t^{\pm 1}\rangle \quad d(t) = 0$

enhancement formal

$\cdots \leftarrow k[\varepsilon]/(\varepsilon^2) \xleftarrow{-1} k[\varepsilon]/(\varepsilon^2) \xleftarrow{1} k[\varepsilon]/(\varepsilon^2) \leftarrow \cdots \longleftarrow\!\shortmid\ t$

$k = \ell[\sqrt{x}] \supset \ell = \mathbb{F}_2(x)$ new non-perfect ground field k/ℓ non-separable extension

2. $\mathcal{T} = \underline{\mathrm{mod}}\, \Gamma \quad \Gamma = \ell\langle \varepsilon, y\rangle/(\varepsilon^2, y^2 + \varepsilon + x, y\varepsilon + \varepsilon y + \varepsilon)$ $c = \Gamma/(\varepsilon) = k$ additive generator

not a k-algebra

$\underline{HH}^{\bullet,*}(\Lambda, \Lambda(\sigma)) = k[t^{\pm 1}, u^{\pm 1}]$

$\Lambda = \underline{\mathrm{End}}(k) = k \Rightarrow \mathcal{T} = \mathrm{mod}\, k$ $\sigma = \mathrm{id}$ $\Lambda(\sigma) = k\langle t^{\pm 1}\rangle$ $|t| = -1$ $|t| = (0, -1)\quad |u| = (1, 0)$

stable Auslander algebra graded Lambda algebra

$A = \mathcal{E}\mathrm{nd}(\cdots \to \Gamma \xrightarrow{\varepsilon} \Gamma \xrightarrow{\varepsilon} \Gamma \to \cdots)$ neither rUMPs $u^3 t$ 0

$A \not\simeq k\langle t^{\pm 1}\rangle$ non-equivalent enhancements

$u\colon \quad k \hookrightarrow k \otimes_\ell k \xrightarrow{\text{mult.}} k \in \mathrm{Ext}^1_{k \otimes_\ell k}(k, k)$

$1 \mapsto \sqrt{x} \otimes 1 + 1 \otimes \sqrt{x}$

Algebraic $\Leftarrow$ Amiot   char $k \neq 2$

B graded algebra, $HH^{\bullet,*}(B,B)$ commutative + shifted Lie algebra

Gerstenhaber relation $[x, y\cdot z] = [x,y]\cdot z + (-1)^{(|x|-1)|y|}\, y\cdot[x,z]$

Massey algebra (B, m) B graded alg. $m \in HH^{3,-1}(B,B)$ UMP $\frac{1}{2}[m,m]=0$

Example $(H^*A, \{m_3\})$ the Massey algebra of A dga

Theorem (lifting units) [M'22] $\eta \in HH^{3,-1}(\Lambda, \Lambda(\sigma))$ unit in $\underline{HH}^{\bullet,*}(\Lambda,\Lambda(\sigma))$

$\Rightarrow \exists!\, m \in HH^{3,-1}(\Lambda(\sigma),\Lambda(\sigma))$ with

- $j^* m = \eta$ rUMP   (mod$_p$ Λ^e, $\otimes_\Lambda$)
- $\frac{1}{2}[m,m]=0$

Example $(\Lambda(\sigma), m)$ the Massey algebra of the Amiot triangulated structure

on proj Λ defined by $\eta : {}_1\Lambda_\sigma \overset{i}{\hookrightarrow} \underbrace{P_3 \to P_2 \to P_1}_{\in \operatorname{proj} \Lambda^e} \overset{p}{\twoheadrightarrow} \Lambda \in \operatorname{Ext}^3_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma) = HH^{3,-1}(\Lambda, \Lambda(\sigma))$

$(HH^{\bullet,*}(B,B), d=[m,-])$ Hochschild-Massey complex of (B,m) $|d|=(2,-1)$

$HH^{\bullet,*}(B,m)$ Hochschild-Massey cohomology

Theorem [M'22] $HH^{p+2,-p}(B,m)=0$ $p>1$ $\Rightarrow \exists!$ A dga $(H^*A, \{m_3\}) \cong (B,m)$

Proposition [M'20] $\cdots \to HH^{p-1,q}(\Lambda,\Lambda(\sigma)) \to HH^{p,q}(\Lambda(\sigma),\Lambda(\sigma)) \to HH^{p,q}(\Lambda,\Lambda(\sigma)) \xrightarrow{\text{id}-\text{conj. by }\sigma} HH^{p,q}(\Lambda,\Lambda(\sigma)) \to \cdots$

Theorem [M'22] $(\Lambda(\sigma),m)$ Massey algebra and j^*m Hochschild-Tate unit

$$\Rightarrow HH^{p+2,q}(\Lambda(\sigma),m)=0 \quad p>1 \quad q\in\mathbb{Z}$$

Idea of proof

Euler derivation $\delta \in HH^{1,0}(B,B)$ $\delta(b)=q\cdot b$, $b\in B^q$ $[\delta,x]=q\cdot x$, $x\in HH^{p,q}(B,B)$

$(HH^{\bullet,*}(\Lambda(\sigma),\Lambda(\sigma)), d=[m,-])$ $\circlearrowleft$ $m\cdot$

endomorphism of complexes

of bidegree $(3,-1)$

bijective on bidegrees (p,q) for $p>1$

$(HH^{\bullet,*}(\Lambda(\sigma),\Lambda(\sigma)), d=[m,-])$ $\circlearrowleft$ $\delta\cdot$

null-homotopy for $m\cdot$

$m\cdot x=[m,\delta]\cdot x=[m,\delta\cdot x]+\delta\cdot[m,x]$

Gerstenhaber relation $x\in HH^{\bullet,*}(\Lambda(\sigma),\Lambda(\sigma))$

Theorem (derived Auslander correspondence) [Π'22] $\quad$ k perfect

We have bijections between equivalence classes of:

$A \mapsto \operatorname{per} A$

1. A dga with additively finite $\operatorname{per} A \ni A$ basic additive generator

quasi-isomorphism

2. $\mathcal{T}$ additively finite algebraic triangulated category

triangulated equivalence

$\operatorname{per} A \mapsto (H^0 A, H^{-1} A)$ $\quad$ Amiot $\uparrow$

$\mathcal{T} \mapsto (\mathcal{T}(c,c), \mathcal{T}(c, c[-1]))$ $\quad$ c additive generator

3. $(\Lambda, {}_1\Lambda_\sigma)$ with
   a) Λ f.d. basic self-injective algebra
   b) ${}_1\Lambda_\sigma$ twisted Λ-bimodule such that $\Omega^3_{\Lambda^e}\Lambda \cong {}_1\Lambda_\sigma$ in $\underline{\operatorname{mod}}\,\Lambda^e$

isomorphisms of (algebra, bimodule in $\underline{\operatorname{mod}}\,\Lambda^e$)

Corollary k arbitrary Λ separable we have bijections between

1. $\{A$ dga with $\operatorname{per} A \simeq \operatorname{proj} \Lambda : A \mapsto \Lambda\} /$ quasi-isomorphism

   $\Lambda(\sigma)$ $d=0$ formal

2. $\{$algebraic triangulated category $\mathcal{T} \simeq \operatorname{proj} \Lambda\} /$ equivalence

   $\mathcal{T} = \operatorname{proj} \Lambda$ $X[-1] = X \otimes_{\Lambda} {}_1\Lambda_\sigma$ exact triangles $\equiv$ split triangles

3. $\{$conjugacy classes in $\operatorname{Out}(\Lambda)\}$

   $\sigma \in \operatorname{Out}(\Lambda)$

Example $\Lambda = k^n$ $\mathcal{T} = (\operatorname{mod} k)^n$ $\operatorname{Out}(\Lambda) = S_n$

$\{$conjugacy classes in $S_n\} \cong \{$integer partitions of $n\}$

the trivial partition corresponds to the only indecomposable triangulated structure, with $\Sigma(X_1 \dots X_n) = (X_2 \dots X_n X_1)$. The rest decompose as the partition.

2. Triangulated categoires with a periodic cluster tilting object

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Definition $d \geq 1$ [Iyama–Yoshino '08]

$\mathcal{T}$ idempotent complete, hom-finite triangulated category

$\cup$

$\mathcal{C}$ functorially finite full subcategory is $d\mathbb{Z}$-cluster tilting if

1. $\mathcal{C} = \{x \in \mathcal{T} \mid \mathcal{T}(x, \mathcal{C}[n]) = 0 \quad \forall\, 0 < n < d\}$

   $= \{x \in \mathcal{T} \mid \mathcal{T}(\mathcal{C}, x[n]) = 0 \quad \forall\, 0 < n < d\}$

2. $\mathcal{C} = \mathcal{C}[d]$

$d = 1$ $\quad \mathcal{C} \subset \mathcal{T}$ $1\mathbb{Z}$-cluster tilting $\Leftrightarrow$ $\mathcal{C} = \mathcal{T}$

If additively finite $\quad \mathcal{C} \xrightarrow{\sim} \operatorname{proj} \Lambda$ f.d. basic self-injective algebra

basic $d\mathbb{Z}$-cluster tilting $\quad c \longmapsto \Lambda \quad$ Lambda algebra

Classify hom-finite algebraic triangulated categories with a $d\mathbb{Z}$-cluster tilting object and their enhancements

Examples (Amiot–Guo–Keller cluster categories) [Keller'08, Amiot'09, Guo'11]

$\mathcal{T} = \mathcal{C}(\Gamma) = \operatorname{per}\Gamma / \{M \text{ with } \sum_{n\in\mathbb{Z}} \dim H^n M\} \ni \Gamma$ $d\mathbb{Z}$-cluster tilting for

1. $\Gamma = \Pi_{d+1}(A)$ derived $(d+1)$-preprojective algebra of
   A d-representation finite algebra ($\operatorname{gldim} A = d$)

   $\Lambda = \Pi_{d+1}(A)$ $(d+1)$-preprojective algebra [Iyama–Oppermann'13]

2. $\Gamma = \Gamma(Q,W)$ completed Ginzburg dga of a quiver Q with
   potential W ($d=2$) provided ... [Keller–Yang'11, Herschend–Iyama'11]

   $\Lambda = J(Q,W)$ the completed Jacobian algebra is f.d. self-injective

Theorem [Wemyss'18] $k=\mathbb{C}$ R isolated compound Du Val singularity (cDV) with crepant resolutions

$$\{\text{resolutions of } \operatorname{spec} R\} \cong \{2\mathbb{Z}\text{-cluster tilting objects in } \underline{\mathrm{CM}}(R)\}$$

In this case Lambda algebras are called contraction algebras (connected)

[Jasso-M'23] k arbitrary

Theorem $\mathcal{T}$ hom-finite algebraic triangulated category
$\cup\!\!\cup$
c basic $d\mathbb{Z}$-cluster tilting object

Λ Lambda algebra of c

If Λ is connected non-separable and Λ/J_Λ is separable TFAE

1. Λ is twisted $(d+2)$-periodic, i.e. $\Omega^{d+2}_{\Lambda^e}\Lambda \cong {}_1\Lambda_\sigma$ in $\underline{\mathrm{mod}}\,\Lambda^e$ for some $\sigma \in \mathrm{Aut}(\Lambda)$

2. $\exists!$ dga A such that $\mathcal{T} \simeq \mathrm{per}\,A : c \mapsto A$

Conjecture [Donovan–Wemyss'16] R_1, R_2 isolated cDV singularities with crepant resolutions and contraction algebras Λ_1, Λ_2

isomorphism $R_1 \cong R_2 \Leftrightarrow D(\Lambda_1) \simeq D(\Lambda_2)$ triangulated equivalence

Proof assembled by Keller

$i=1,2$ $\begin{cases} \Lambda_i \text{ contraction algebra of } c_i \in \underline{CM}(R_i) \ \ 2\mathbb{Z}\text{-cluster tilting object} \\ A_i \text{ dga with } \operatorname{per} A_i \simeq \underline{CM}(R_i) : A_i \mapsto c_i \end{cases}$ $\Rightarrow$ $H^0 A_i = \Lambda_i$ connected

$\Rightarrow$ [Wemyss'18, Dugas'15]

$\Leftarrow$ $D(\Lambda_1) \simeq D(\Lambda_2) \Rightarrow$ we can assume $\Lambda_1 \cong \Lambda_2$ changing c_2 [August'20]

$\Rightarrow A_1 \simeq A_2$ [Jasso–M.'23] $\Rightarrow R_1 \cong R_2$ [Hua–Keller'18]

The only separable case is $\Lambda = \mathbb{C}$ $\square$

Theorem (derived Auslander–Iyama correspondence) [Jasso–M '23] k perfect

We have bijections between equivalence classes of:

A

$\downarrow$

1. A dga with hom-finite $\operatorname{per} A \ni A$ basic $d\mathbb{Z}$-cluster tilting

   quasi-isomorphism

$(\operatorname{per} A, A)$ $\qquad$ $(\mathcal{T}, c)$

2. $(\mathcal{T}, c)$ with

   a) $\mathcal{T}$ hom-finite algebraic triangulated category

   no Amiot

   b) $c \in \mathcal{T}$ basic $d\mathbb{Z}$-cluster tilting

   triangulated equivalence preserving the object

$\downarrow$ $\qquad$ $\downarrow$

$(H^0A, H^{-d}A)$ $\qquad$ $(\mathcal{T}(c,c), \mathcal{T}(c, c[-d]))$

3. $(\Lambda, {}_1\Lambda_\sigma)$ with

   a) Λ f.d. basic self-injective algebra

   b) ${}_1\Lambda_\sigma$ twisted Λ-bimodule such that $\Omega^{d+2}_{\Lambda^e}\Lambda \cong {}_1\Lambda_\sigma$ in $\underline{\operatorname{mod}}\,\Lambda^e$

   isomorphisms of (algebra, bimodule in $\operatorname{mod} \Lambda^e$)

Definition [Geiss-Keller-Opperman '13]

A $(d+2)$-angulated category consists of

1. $\mathcal{C}$ additive category
2. $\Sigma : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ self-equivalence d-shift
3. $X_1 \to X_2 \to \cdots \to X_{d+2} \to \Sigma X_1$ exact $(d+2)$-angles

\+ axioms like

$\forall$ object $X \xrightarrow{1} X \to 0 \to \cdots \to 0 \to \Sigma X$ exact

$\forall$ map $Y \xrightarrow{f} Z \to X_3 \to \cdots \to X_{d+2} \to \Sigma Y$ exact

$\forall$ square
$$\begin{array}{ccccccccccc} X_1 & \to & X_2 & \to & X_3 & \to & \cdots & \to & X_{d+2} & \to & \Sigma X_1 \text{ exact} \\ \downarrow g & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \Sigma g \\ Y_1 & \to & Y_2 & \to & Y_3 & \to & \cdots & \to & Y_{d+2} & \to & \Sigma Y_1 \text{ exact} \end{array}$$

$d=1$ 3-angulated $\equiv$ triangulated

Theorem $\mathcal{T}$ idempotent complete hom-finite triangulated category TFAE

1. $\mathcal{C} \subset \mathcal{T}$ $d\mathbb{Z}$-cluster tilting

2. a) $\mathcal{T} = \mathrm{thick}(\mathcal{C})$ and $\mathcal{C}[d] = \mathcal{C}$

   b) $\mathcal{T}(\mathcal{C}, \mathcal{C}[n])$ $\forall\, 0 < n < d$ d-rigidity

   c) $\mathcal{C}$ $(d+2)$-angulated with $\Sigma = [d]$ and exact $(d+2)$-angles

1. $\Longrightarrow$ 2. [Geiss - Keller - Opperman '13]

1. $\Longleftarrow$ 2. [Jasso - M. '23]

These $(d+2)$-angulated categories are called normal

Theorem [Lin'19] Amiot-Lin $(d+2)$-angulated structure on $\operatorname{proj}\Lambda$ induced by

$$\boxed{{}_1\Lambda_\sigma \overset{i}{\hookrightarrow} \underbrace{P_{d+2}\to\cdots\to P_2\to P_1}_{\in \operatorname{proj}\Lambda^e} \overset{p}{\twoheadrightarrow} \Lambda} \in \operatorname{Ext}^{d+2}_{\Lambda^e}(\Lambda,{}_1\Lambda_\sigma) = HH^{d+2}(\Lambda,{}_1\Lambda_\sigma)$$

- d-shift $\Sigma^{-1}X = X\otimes_\Lambda {}_1\Lambda_\sigma$
- exact $(d+2)$-angles $M\otimes_\Lambda (P_1\otimes_\Lambda {}_1\Lambda_\sigma \to P_{d+2}\to\cdots\to P_2\to P_1)$ $M\in\operatorname{mod}\Lambda$

  $P_1\otimes_\Lambda {}_1\Lambda_\sigma \to {}_1\Lambda_\sigma \xrightarrow{i} P_{d+2}$

  meta-$(d+2)$-angle

Example [Erdmann-Skowroński'08, Dugas'10]

Λ self-injective algebra of finite representation type

Algebraic normal
$\Rightarrow$ Amiot–Lin

$$\mathcal{T} \simeq \operatorname{per} A \;:\; c \mapsto A$$ enhancement

$$d\mathbb{Z}\text{-cluster tilting} \longrightarrow \mathcal{C} \simeq \operatorname{proj}\Lambda \;:\; c \mapsto \Lambda$$ Lambda algebra

$X[-d] = X \otimes_\Lambda {}_1\Lambda_\sigma$ $\quad X \in \operatorname{proj}\Lambda$ $\quad \sigma \in \operatorname{Aut}(\Lambda)$

d-sparse graded Lambda algebra

$$H^*A = \bigoplus_{n\in\mathbb{Z}} \mathcal{T}(c, c[n]) = \bigoplus_{dn\in d\mathbb{Z}} {}_1\Lambda_\sigma^{\otimes_\Lambda(-n)} = \bigoplus_{dn\in d\mathbb{Z}} {}_1\Lambda_{\sigma^{-n}} = \Lambda(\sigma,d) = \frac{\Lambda\langle t^{\pm 1}\rangle}{(\lambda t - t\sigma(\lambda))_{\lambda\in\Lambda}} \qquad |t| = -d$$

minimal model $(\Lambda(\sigma,d), m_{d+2}, m_{2d+2}, \dots, m_{dn+2}, \dots)$ d-sparse A_∞-algebra

$j : \Lambda \hookrightarrow \Lambda(\sigma,d)$ inclusion of degree 0 part

$$j^* : HH^{d+2,-d}(\Lambda(\sigma,d), \Lambda(\sigma,d)) \longrightarrow HH^{d+2,-d}(\Lambda, \Lambda(\sigma,d)) = HH^{d+2}(\Lambda, {}_1\Lambda_\sigma) = \operatorname{Ext}^{d+2}_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma)$$

$(d+2)$-UMP $\quad \{m_{d+2}\} \longmapsto j^*\{m_{d+2}\}$ $\quad (d+2)$-rUMP

The normal $(d+2)$-angulated structure on $\operatorname{proj}\Lambda$ is Amiot's for the rUMP $j^*\{m_{d+2}\}$

$j^*\{m_{d+2}\}$ represented by ${}_1\Lambda_\sigma \overset{i}{\hookrightarrow} \underbrace{P_{d+2} \to \cdots \to P_2 \to P_1}_{\in \operatorname{proj}\Lambda^e} \overset{p}{\twoheadrightarrow} \Lambda \iff j^*\{m_{d+2}\}$ unit in $\underline{HH}^{\bullet,*}(\Lambda, \Lambda(\sigma,d))$

Example (Reid's pagoda) $k = \mathbb{C}$ $d = 2$

$\mathcal{T} = \underline{\mathrm{CM}}(R)$ $R = k[[u,v,x,y]]/(uv + x^2 - xy^3)$

$c = (u,x)$ basic $2\mathbb{Z}$-c.t. cDV singularity

$\Lambda \cong k[\varepsilon]/(\varepsilon^3)$ stable Auslander algebra

$\sigma = \mathrm{id}$

$\Lambda(\sigma,d) = \Lambda[t^{\pm 1}]$ $|t| = -2$ graded Auslander algebra

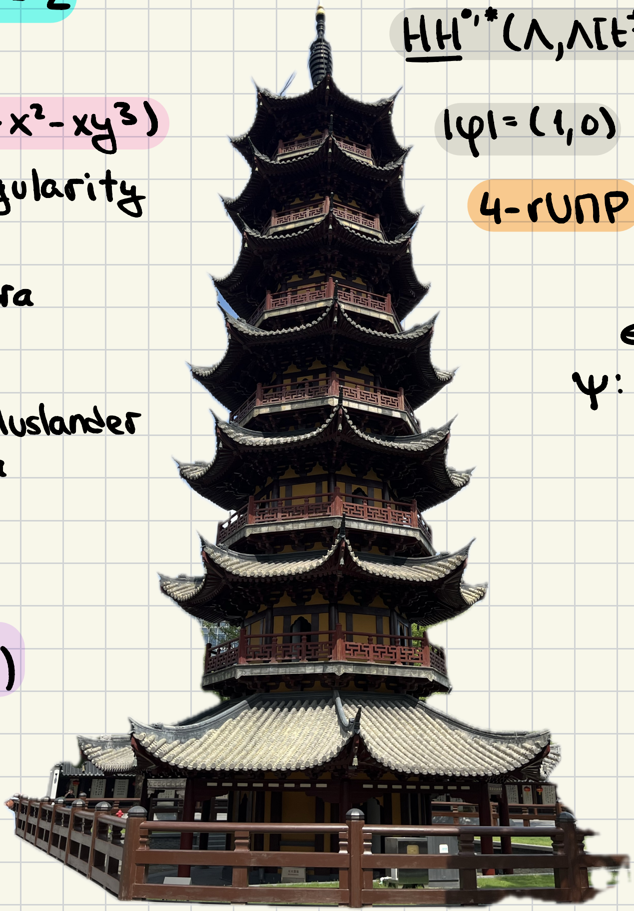
$S = k[[x,y]]/(x^2 - xy^3)$

$A = \mathcal{E}\mathrm{nd}(\cdots \to S \xrightarrow{x} S \xrightarrow{x-y^3} S \xrightarrow{x} S \to \cdots)$

ε: y y y y

$\varepsilon^3 = 0$: $\cdots \to S \xleftarrow{1} S \xleftarrow{-1} S \xleftarrow{1} S \to \cdots$

enhancement by Knörrer periodicity



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$\underline{HH}^{\bullet,*}(\Lambda, \Lambda[t^{\pm 1}]) = k[\varphi, \psi^{\pm 1}, \varepsilon, t^{\pm 1}]/(\varepsilon^2)$

$|\varphi| = (1,0)$ $|\psi| = (2,0)$ $|\varepsilon| = (0,0)$

4-rUnP $\psi^2 t$ $|t| = (0,-2)$

$\in \mathrm{Ext}^2_{\Lambda^e}(\Lambda,\Lambda)$

$\psi: \Lambda \hookrightarrow \Lambda\otimes\Lambda \to \Lambda\otimes\Lambda \overset{\text{mult.}}{\twoheadrightarrow} \Lambda$

$1 \mapsto \varepsilon\otimes 1 - 1\otimes\varepsilon$

$1 \mapsto \varepsilon^2\otimes 1 + \varepsilon\otimes\varepsilon + 1\otimes\varepsilon^2$

Minimal model

$(\Lambda[t^{\pm 1}], m_4, 0 \ldots)$

2-sparse A_∞-algebra

computed in joint discussions with Booth and Wemyss

Corollary k arbitrary Λ separable we have bijections between

1. $\{ A$ dga with $H^0 A \cong \Lambda$ hom-finite per A
   $d\mathbb{Z}$-cluster tilting $A \in \operatorname{per} A \} /$ quasi-isomorphism

   $\Lambda(\sigma, d)$ $\partial = 0$ formal

2. $\{$ hom-finite algebraic triangulated category $\mathcal{T} \ni c$ $d\mathbb{Z}$-cluster tilting object with $\mathcal{T}(c,c) \cong \Lambda \} /$ triangulated equivalence preserving the object

   $\mathcal{T} = (\operatorname{proj} \Lambda)^d \ni c = (\Lambda\, 0 \dots 0)$

   $\Sigma^{-1}(X_1 \dots X_d) = (X_2 \dots X_d\, X_1 \otimes_\Lambda {}_1\Lambda_\sigma)$ exact triangles $\equiv$ split triangles

3. $\{$ conjugacy classes in $\operatorname{Out}(\Lambda) \}$ independent of d

   $\sigma \in \operatorname{Out}(\Lambda)$

3. The Calabi-Yau property

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B graded algebra

DB graded dual B-bimodule $(DB)^i = \mathrm{Hom}_k(B^{-i}, k)$

B is (bimodule right) n-Calabi-Yau (n-CY) if $B[n] \cong DB$ as B-bimodules

B n-CY $\Rightarrow$ Batalin-Vilkovisky operator $\Delta : HH^{p,q}(B,B) \longrightarrow HH^{p-1,q}(B,B)$

$$\Delta(x \cdot y) = \Delta(x) \cdot y + (-1)^{|x|} x \cdot \Delta(y) + [x,y], \quad \Delta^2 = 0.$$

The n-CY definition extends to graded categories and to dgas.

A triangulated category $\mathcal{T}$ is n-CY if the associated graded category $\mathcal{T}^\bullet$, with the same objects and $\mathcal{T}^i(x,y) = \mathcal{T}(x, y[i])$, is.

A n-CY dga $\Rightarrow$ per $A = \mathcal{T}$ n-CY $\Rightarrow$ H^*A, $\mathcal{T}^\bullet(x,x)$, $x \in \mathcal{T}$, n-CY.

Examples (Amiot–Guo–Keller cluster categories)

$\mathcal{T} = \mathcal{C}(\Gamma)$ is d-CY for

1. $\Gamma = \Pi_{d+1}(A)$ derived $(d+1)$-preprojective algebra of A d-representation finite algebra
2. $\Gamma = \Gamma(Q, W)$ completed Ginzburg dga of a quiver Q with potential W $(d=2)$ with $J(Q,W)$ f.d. self-injective

[Jasso–M.]

Theorem A dga with hom-finite $\operatorname{per} A \ni A$ basic $d\mathbb{Z}$-cluster tilting A d-CY dga $\Leftrightarrow$ $H^*A \cong \Lambda(\sigma, d)$ d-CY graded algebra and $\boxed{\Delta(\{m_{d+2}\}) = 0}$

Proof Homotopy theory of operads, their algebras and their bimodules

This equation holds in all examples that we have been able to compute

4. Open problems

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Λ f.d. algebra which is $\begin{cases} \bullet \text{ basic} \\ \bullet \text{ connected} \\ \bullet \text{ non-separable} \\ \bullet \text{ twisted } (d+2)\text{-periodic} \end{cases}$

Find explicitly:

1. A dga with hom-finite $\operatorname{per} A \ni A$ $d\mathbb{Z}$-cluster tilting and $H^0 A = \Lambda$
2. $(\Lambda(\sigma,d), \boxed{m_{d+2}}, m_{2d+2}, \dots, m_{nd+2}, \dots)$ minimal model of A (A_∞-algebra)
3. Down-to-earth description of $\mathcal{T} = \operatorname{per} A$
4. d-CY structure on A or $(\Lambda(\sigma,d), \{m_{nd+2}\}_{n\geq 1})$ when $\Lambda(\sigma,d)$ is d-CY

   and $\boxed{\Delta(\{m_{d+2}\}) = 0}$
5. Does this equation always hold? Proof or counterexample

THANKS A LOT