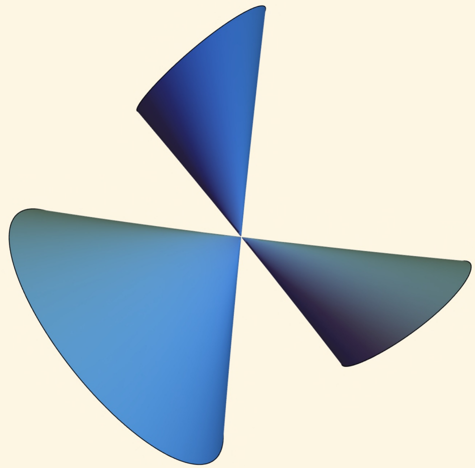

An application of homotopy theory to singularity theory.

Fernando Muro (Sevilla)

joint work w/ G. Jasso (Lund) & B. Keller (Paris-Jussieu)

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R local noetherian ring

$D^b(R)$ derived category of bounded complexes of finitely generated R-modules

$\cup$

$D^c(R) = \mathrm{Perf}(R)$ same as above ... projective R-modules

compact perfect

$$R \text{ regular} \iff D^c(R) \xrightarrow{\sim} D^b(R)$$

$D^{sg}(R) := D^b(R) / D^c(R)$ derived category of singularities

Verdier quotient

Hypersurface in $\mathbb{C}^4$ (germ at the origin)

$R = \mathbb{C}[[u,v,x,y]]/(h)$ h non-constant polynomial

$D^{sg}(R)$ has many good properties: it's rather small in all possible ways

- $\dim_{\mathbb{C}} \mathrm{Hom}_{D^{sg}(R)}(X,Y) < \infty \quad \forall X,Y$ [Buchweitz '86, Yoshino '90]

- $\Sigma^2 X \cong X$ naturally $\forall X \in D^{sg}(R)$ [Eisenbud '80]

- $\mathrm{Hom}_{D^{sg}(R)}(X,Y)^* \cong \mathrm{Hom}_{D^{sg}(R)}(Y, \Sigma^2 X)$ [Auslander '78] naturally $\forall X,Y$

Compound Du Val singularity (cDV)

$$h = f(u,v,x) + y \cdot g(u,v,x,y)$$

with $\mathbb{C}[[u,v,x]]/(f)$ a Du Val singularity

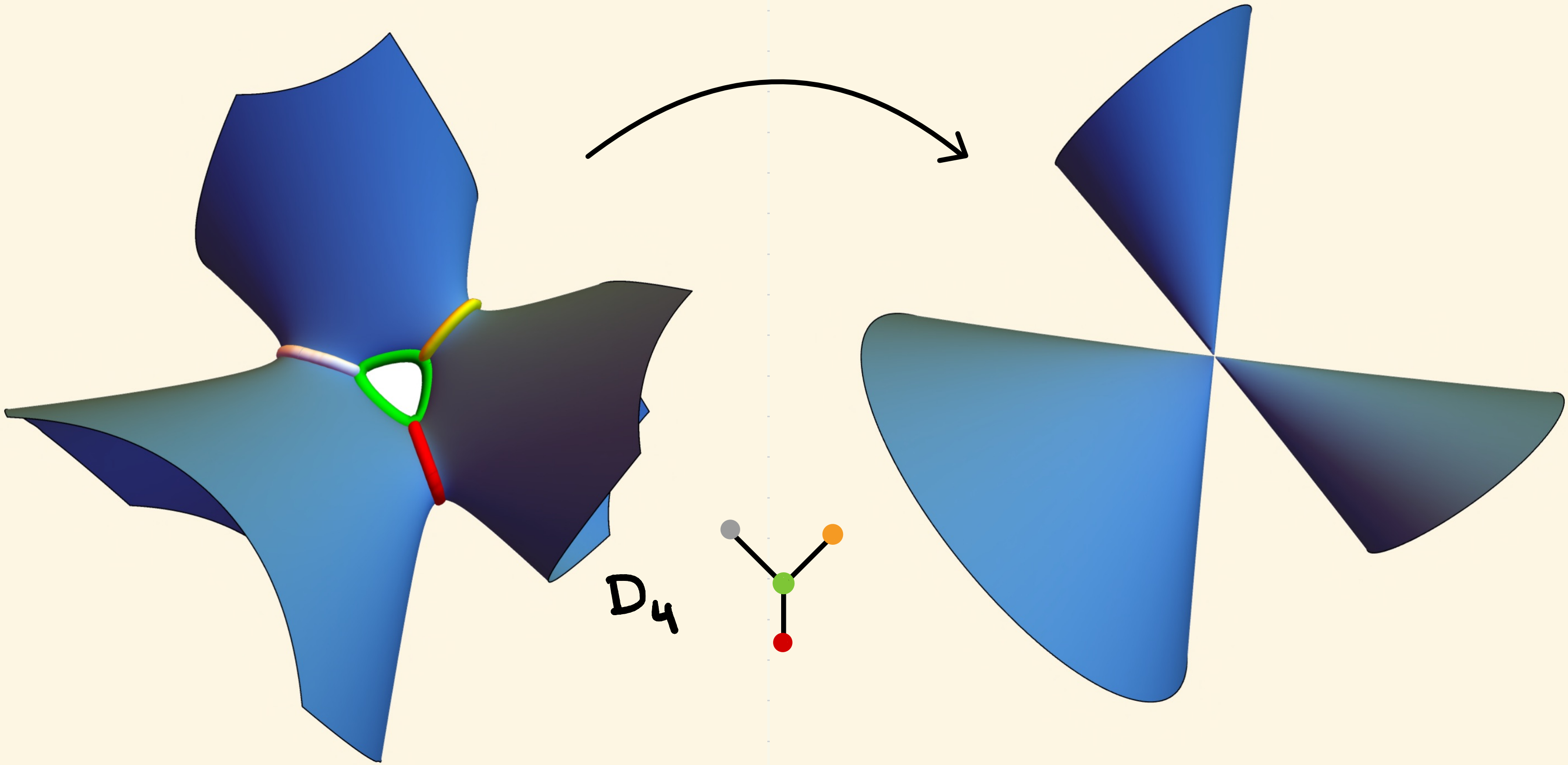
$$\mathbb{C}^2/G \qquad G \subset SL_2(\mathbb{C}) \qquad |G| < \infty$$

The latter are classified by Dynkin diagrams of type A_n, D_n, E_6, E_7, E_8.

The former are not classified. We will assume they are isolated and have smooth minimal models.

in the sense of algebraic geometry

Du Val singularity resolution



$X = \mathrm{Spec}(R)$ cDV

[Wemyss'18]

Minimal models $\longleftrightarrow$ basic $2\mathbb{Z}$-cluster tilting object

$Y \longrightarrow X$ $\qquad$ $T \in D^{sg}(R)$

The contraction algebra of a minimal model

$$\Lambda := \mathrm{End}_{D^{sg}(R)}(T)$$ finite-dimensional $\mathbb{C}$-algebra

Conjecture: [Donovan-Wemyss'16]

Let $X_1 = \mathrm{Spec}(R_1)$ and $X_2 = \mathrm{Spec}(R_2)$ be cDVs equipped with minimal models

$$D(\Lambda_1) \simeq D(\Lambda_2) \iff R_1 \cong R_2 \text{ i.e. } X_1 \cong X_2$$

$\Leftarrow$ Follows from [Dugas'15, Wemyss'18]

Theorem: [August '20]

The Donovan–Wemyss conjecture is equivalent to

$$\Lambda_1 \cong \Lambda_2 \implies R_1 \cong R_1$$

Since the contaction algebras of a given cDV form a whole derived equivalence class of finite-dimensional basic algebras.

The derived contraction algebra of a minimal model

$$\Lambda := \mathbb{R}\mathrm{End}_{D^{sg}(R)}(T)$$

It is a differential graded algebra (DGA) with

$$H^0(\Lambda) = \Lambda, \qquad H^*(\Lambda) = \Lambda[t^{\pm 1}], \quad |t| = 2.$$

concentrated in $2\mathbb{Z}$

Theorem: [Hua-Keller'23]

$$\Lambda_1 \simeq \Lambda_2 \implies R_1 \cong R_2$$

Corollary: The Donovan-Wemyss conjecture is equivalent to

$$H^*(\Lambda_1) \cong H^*(\Lambda_2) \Leftrightarrow \Lambda_1 \cong \Lambda_2 \implies \Lambda_1 \simeq \Lambda_2$$

When is a DGA determined by its cohomology?

A DGA A is formal if $A \simeq H^*(A)$

A graded algebra B is intrinsically formal if

$$H^*(A) \cong B \implies A \text{ is formal}$$

Theorem: [Kadeishvili '88] Intrinsic formality criterion

$$HH^{n,2-n}(B) = 0, \ n \geq 2 \implies B \text{ intrinsically formal}$$

Theorem: R cDV with contraction algebra Λ, TFAE:

a) $\Lambda[t^{\pm 1}]$, $|t| = 2$, intrinsically formal

b) $\Lambda = \mathbb{C}$

c) $R = \mathbb{C}[[u,v,x,y]]/(uv-xy)$

d) the minimal model is the Atiyah flop

k perfect ground field (e.g. $k = \bar{k}$ or $|k| < \infty$)

$\mathcal{T}$ algebraic triangulated category (closed under retracts) with $\dim_k \operatorname{Hom}_{\mathcal{T}}(X, Y) < \infty$ $\forall X, Y \in \mathcal{T}$

A basic object $T \in \mathcal{T}$ is $d\mathbb{Z}$-cluster tilting (CT) if:

- $\operatorname{Hom}_{\mathcal{T}}(T, \Sigma^i T) = 0$ if $i \notin d\mathbb{Z}$ gaps of length $d-2$
- $\Sigma^d T \cong T$ d-periodicity
- $\forall X \in \mathcal{T}$ $\exists$ $C_i \longrightarrow Y_{i-1} \longrightarrow Y_i \longrightarrow \Sigma C_i$ exact triangle $1 \leq i \leq d$, such that:

  $\mathcal{T} = \langle T \rangle$

  - $Y_0 = 0$ T is like a 0-sphere
  - $Y_d = X$ X is like a connected CW-complex of dim $= d-1$
  - C_i is a retract of $\Sigma^{i-2}(T \oplus \overset{n}{\cdots} \oplus T)$ for some $n \geq 0$

Example: $\mathcal{T} = D^c(K(n)) \ni T = K(n)$ mod p Morava K-theory is $d\mathbb{Z}$-CT for $d = 2(p^n - 1)$

If $\mathcal{T}$ is algebraic and $T \in \mathcal{T}$ is $d\mathbb{Z}$-CT and $\mathbb{\Lambda} := \mathbb{R}\mathrm{End}_{\mathcal{T}}(T)$ derived Lambda

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\sim} & D^c(\mathbb{\Lambda}) \\ T & \longmapsto & \mathbb{\Lambda} \end{array}$$

Moreover

concentrated in $d\mathbb{Z}$ graded Lambda

$$H^*(\mathbb{\Lambda}) = \frac{\Lambda\langle t^{\pm 1}\rangle}{(t\lambda - \sigma(\lambda)t)} =: \Lambda(\sigma, d), \quad |t| = -d, \quad \sigma \in \mathrm{Aut}(\Lambda).$$

where $\Lambda := H^0(\mathbb{\Lambda})$ and

choice of $f\colon \Sigma^d T \xrightarrow{\cong} T$

$$\begin{array}{ccccc} \sigma\colon \Lambda = \mathrm{End}_{\mathcal{T}}(T) & \underset{\cong}{\xrightarrow{\Sigma^d}} & \mathrm{End}_{\mathcal{T}}(\Sigma^d T) & \cong & \mathrm{End}_{\mathcal{T}}(T) = \Lambda \\ g & \longmapsto & \Sigma^d g & \longmapsto & f \circ (\Sigma^d g) \circ f^{-1} \end{array}$$

$[\sigma] \in \mathrm{Out}(\Lambda)$ is well defined and determines $\Lambda(\sigma, d)$ up to iso.

An A_∞-algebra $(A, m_1, m_2, m_3, \dots)$ is a graded vector space A equipped with maps

$$m_n : A \otimes \overset{n}{\cdots} \otimes A \longrightarrow A \qquad |m_n| = 2-n$$

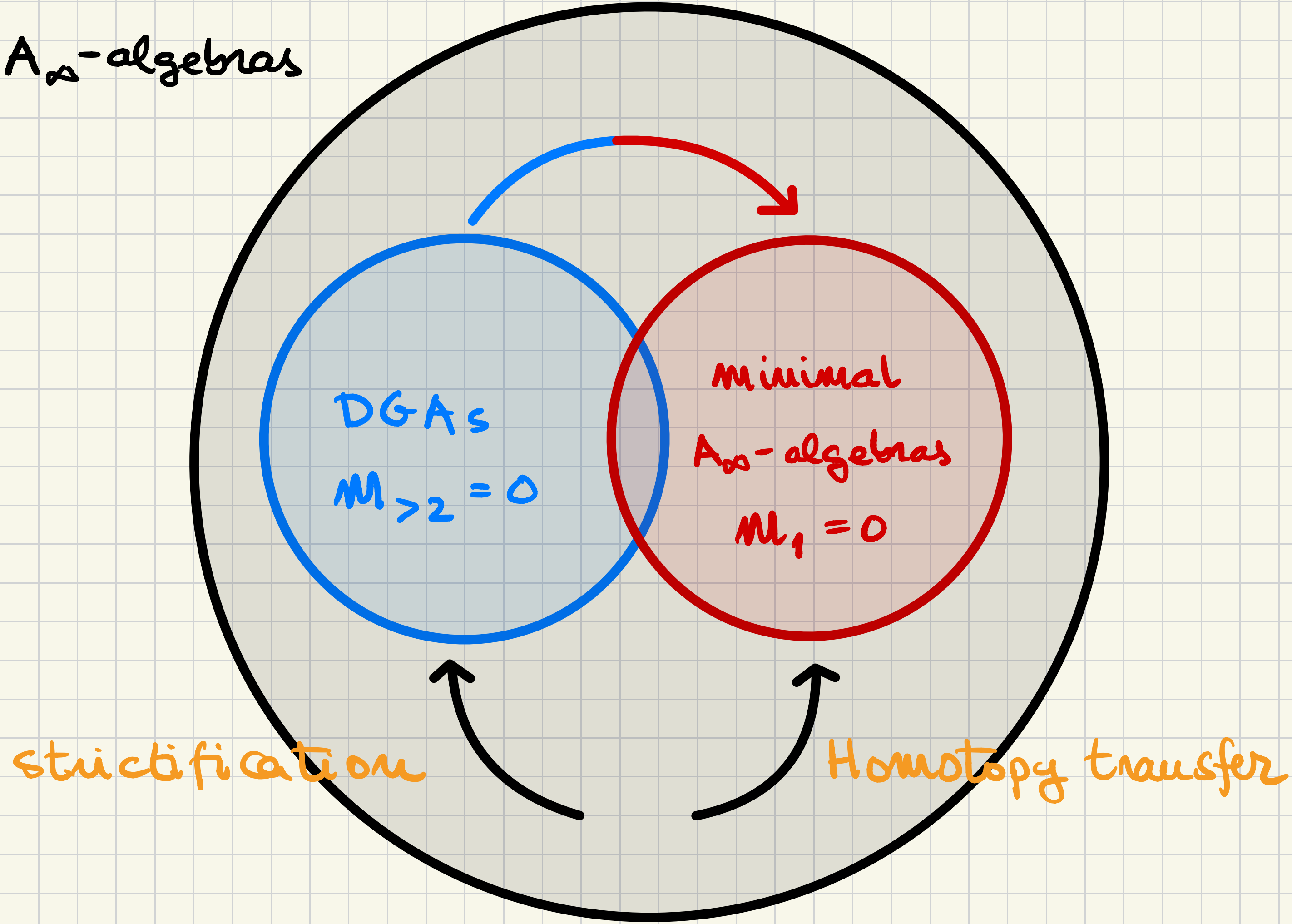
such that:

- m_1 is a differential $\quad m_1^2 = 0$ $\qquad xy = m_2(x,y)$
- m_1 satisfies the Leibniz rule w.r.t. the product m_2
- m_2 is homotopy associative w/ cochain homotopy m_3

  $x(yz) = (xy)z$

...

In particular $H^*(A)$ is a graded algebra with product induced by m_2.



A_∞-algebras
DGAs
$m_{>2}=0$
minimal
A_∞-algebras
$m_1=0$
strictification
Homotopy transfer

in the sense of homotopy theory

The **minimal model** of a DGA A with $H^*(A)$ concentrated in $d\mathbb{Z}$ looks like

$$(\underbrace{H^*(A), m_2}_{\text{graded algebra}}, m_{d+2}, m_{2d+2}, m_{3d+2}, \dots)$$

The **universal Massey product of length $d+2$ ($(d+2)$-UMP)** is
or Toda bracket

$$\{m_{d+2}\} \in HH^{d+2,-d}(H^*(A))$$ well defined by A

For $d=1$ [Baues-Dreckmann'89, Benson-Krause-Schwede'04]

Proposition:

If A is formal $\Rightarrow \{m_{d+2}\} = 0$

A **d-Massey algebra** (B, m) is a graded algebra B concentrated in $d\mathbb{Z}$ equipped with

$$m \in HH^{d+2,-d}(B)$$

such that:

$$\frac{1}{2}[m, m] = 0 \qquad \text{Gerstenhaber Lie bracket}$$

Example: $(H^*(A), \{m_{d+2}\})$ A DGA w/ $H^*(A)$ concentrated in $d\mathbb{Z}$

part of the data

$$\{\text{DGAs } A \text{ w/ } H^*(A) \cong B\} \longrightarrow HH^{d+2,-d}(B)$$
$$A \longmapsto \{m_{d+2}\}$$

Source and target carry a compatible action of $\mathrm{Aut}(B)$.

All DGAs in an orbit are quasi-isomorphic.

A d-Massey algebra (B, m) is intrinsically formal if, given DGAs A_1, A_2

$$\begin{array}{ccccc} H^*(A_1) & \cong & H^*(A_2) & \cong & B \\ \{m^{A_1}_{d+2}\} & \mapsto & \{m^{A_2}_{d+2}\} & \mapsto & m \end{array} \implies A_1 \simeq A_2$$

induced isomorphisms in Hochschild cohomology

The Hochschild cohomology of a d-Massey algebra (B, m)

$$HH^{*,*}(B, m)$$

is the cohomology of the complex

Gerstenhaber Lie bracket again

$$(HH^{*,*}(B), [m, -])$$

Theorem: Intrinsic formality criterion for d-Massey algebras

$$HH^{n+2,-n}(B, m) = 0 \text{ for } n > d \implies (B, m) \text{ intrinsically formal}$$

Main results

Theorem:

Given two DGAs Λ_1, Λ_2 such that $\Lambda_1 \in D^c(\Lambda_1)$ and $\Lambda_2 \in D^c(\Lambda_2)$ are $d\mathbb{Z}$-CT

$$H^*(\Lambda_1) \cong H^*(\Lambda_2) \implies \Lambda_1 \simeq \Lambda_2 .$$

Corollary:

The Donovan-Wemyss conjecture holds.

Proof:

The proof consists of two steps:

Let $\Lambda(\sigma,d) \cong H^*(\Lambda_1) \cong H^*(\Lambda_2)$.

1. Show that the $(d+2)$-UMPs of Λ_1 and Λ_2

$$\{m_{d+2}^{\Lambda_1}\}, \{m_{d+2}^{\Lambda_2}\} \in HH^{d+2,-d}(\Lambda(\sigma,d))$$

belong to the same $\mathrm{Aut}(\Lambda(\sigma,d))$-orbit.

2. Show that the Hochschild cohomology of the d-Massey algebras of Λ_1 and Λ_2 satisfy

$$HH^{n+2,*}(\Lambda(\sigma,d),\{m_{d+2}^{\Lambda_i}\}) = 0, \quad n > d, \; i = 1,2.$$

$\square$

What's behind the intrinsic formality criteria?

B graded algebra concentrated in $d\mathbb{Z}$

$X_\infty :=$ classification space of DGAs A w/ $H^*(A) \cong B$

$= \lim\limits_n X_n$ homotopy limit of a tower of fibrations

$X_n :=$ classification space of A_{n+2}-algebras A w/ $H^*(A) = B$

$0 \leq n \leq \infty$ truncated A_∞-algebras

Bousfield-Kan spectral sequence $E_r^{pq} \Rightarrow \pi_{q-p}(X_\infty)$

Base point a DGA A. It carries obstructions by [Bousfield'89]

$E_2^{pq} = HH^{p+1,-q}(B)$ $\Rightarrow$ [Kadeishvili'88]

mostly

$E_3^{pq} = HH^{p+1,-q}(B, \{m_{d+2}\})$ $\Rightarrow$ Intrinsic formality criterion for d-Massey algebras.

mostly

Bibliography

THE DERIVED AUSLANDER–IYAMA CORRESPONDENCE

GUSTAVO JASSO AND FERNANDO MURO

With an appendix by Bernhard Keller

Abstract. We work over a perfect field. Recent work of the second-named author established a Derived Auslander Correspondence that relates finite-dimensional self-injective algebras that are twisted 3-periodic to algebraic triangulated categories of finite type. Moreover, the aforementioned work also

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THE DONOVAN–WEMYSS CONJECTURE VIA THE TRIANGULATED AUSLANDER–IYAMA CORRESPONDENCE

GUSTAVO JASSO, BERNHARD KELLER, AND FERNANDO MURO

Abstract. We provide an outline of the proof of the Donovan–Wemyss Conjecture in the context of the Homological Minimal Model Program for threefolds. The proof relies on results of August, of Hua and the second-named au-

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