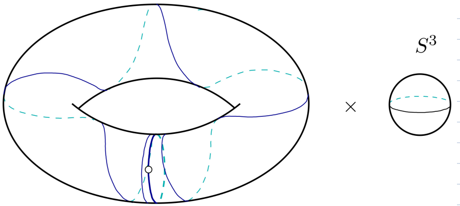

Hypercommutative algebras and differential forms

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Operadic Methods in Geometry

U. Barcelona, 22–24 November 2023

Poisson manifold (M, π)

$\pi \in \Gamma(\Lambda^2 T(M))$ s.t. $[\pi, \pi] = 0$ in $\Gamma(\Lambda^* T(M))$

bivector field

Gerstenhaber algebra of polyvector fields

Gerstenhaber $\equiv$ commutative + Lie + compatibility

$[a, bc] = [a,b]c \pm b[a,c]$

The interior product (contraction)

$$i_\pi : \Omega(M) \longrightarrow \Omega(M)$$

$\Omega(M) = \Gamma(\Lambda^* T^*(M))$

de Rham complex (commutative algebra) differential forms

is a differential operator of degree -2 and order ≤ 2

A commutative algebra over a field k of char $k = 0$
M, N A-modules

Differential operator $\phi : M \longrightarrow N$ of order $\leq n$:

$n = -1$ $\quad \phi = 0$

$n \geq 0$ $\quad$ the operator $[\phi, a] : A \longrightarrow A$
$$x \longmapsto \phi(ax) - a\phi(x)$$
is a differential operator of order $\leq n-1$ $\forall a \in A$

Example: • $n = 0$ $\quad$ A-module morphism
• $\phi : A \longrightarrow A$ unital + Leibniz $\Rightarrow$ order ≤ 1
$\Leftarrow$
if $\phi(1) = 0$ normalization

(M,π) Poisson manifold

exterior differential

$$\Delta := [i_\pi, d] = i_\pi d - d\, i_\pi : \Omega(M) \longrightarrow \Omega(M)$$

Batalin – Vilkovisky operator

is a differential operator of degree -1 and order ≤ 2

Batalin – Vilkovisky (BV) algebra

$\equiv$ commutative algebra

+ differential operator Δ of degree -1 and order ≤ 2

+ $\Delta^2 = 0$

Example: $\Omega(M)$

Koszul bracket

$$[x,y] = \Delta(xy) - \Delta(x)y - (-1)^{|x|} x\,\Delta(y)$$

deviation from Leibniz rule

$\Omega(M)$ Gerstenhaber algebra

$$\underset{[d,\Delta]}{\overset{}{d\Delta + \Delta d}} = 0 \rightsquigarrow \Delta : H^*(M) \longrightarrow H^*(M)$$ de Rham cohomology

$$[-,-] : H^*(M) \otimes H^*(M) \longrightarrow H^*(M)$$ Lie algebra

But $\Delta = 0$ on $H^*(M) \Rightarrow H^*(M)$ abelian Lie algebra

Theorem [Sharygin and Talalaev '08]

The Lie algebra $\Omega(M)$ is formal

Corollary

The Lie algebra $\Omega(M)$ is homotopy abelian

in the L_∞ sense

What can we say about $\Omega(M)$ as a BV-algebra?

(It's not formal, not even as a commutative algebra)

To which extent is Δ homotopically trivial?

Operads

$$
\begin{tikzcd}
\underset{\text{dual numbers}}{k[\Delta]/(\Delta^2)} \arrow[rr, "\text{inclusion of arity 1 part}"] \arrow[d, "\text{augmentation}"'] \arrow[drr, phantom, "\text{homotopy push-out}"] & & BV \arrow[d] \\
k \arrow[rr] & & \boxed{BV/\Delta}
\end{tikzcd}
$$

aug. ideal (Δ)

Def: trivialized BV-operad

[Drummond-Cole – Vallette '13,
Khoroshkin – Markarian – Shadrin '13,
Dotsenko – Shadrin – Vallette '15]

Theorem:

BV/Δ is formal and $H^*(BV/\Delta) = $ Hypercom

Koszul operad

Theorem:

$\Omega(M)$ is a BV/Δ – algebra for (M,π) Poisson manifold and hence a hypercommutative algebra

Corollary:

$H^*(M)$ is an H_∞ – algebra ∞-quasi-isomorphic to $\Omega(M)$

Hypercommutative algebra A:

$m_n : A \otimes \overset{n}{\cdots} \otimes A \longrightarrow A \qquad n \geq 2$

$|m_n| = 2(2-k)$

m_n totally symmetric

such that for each $a \in A$ the operation

$\mu_a : A \otimes A \longrightarrow A \qquad \mu_a(x,y) := \sum_{n=2}^{\infty} \frac{1}{(n-2)!} m_n(x,y,a,\dots,a)$

is associative. In particular (A, m_2) is the underlying commutative algebra.

BV/Δ - algebra A is a BV-algebra equipped with:

$$\phi_n : A \longrightarrow A \qquad n \geqslant 1 \qquad |\phi_n| = -2n$$

such that

$$[\phi_1, d] = \Delta$$

$$\sum_{i=1}^{n} \sum_{j_1+\cdots+j_i=n} \frac{1}{i!} [\phi_{j_1}, [\phi_{j_2}, \cdots [\phi_{j_i}, d] \cdots] = 0 \qquad n \geqslant 2$$

Example: $\Omega(M)$ for (M,π) Poisson

$$\phi_1 = i_\pi \quad \text{and} \quad \phi_n = 0, \quad n \geqslant 2$$

Operadic viewpoint

$$
\begin{tikzcd}
 & & \mathrm{Gerst} \arrow[d] & & \\
\mathrm{Comm} \arrow[urr, bend left, "\text{inclusion}"] \arrow[dr, "\text{inclusion}"] \arrow[dddrr, bend right, "\text{Exterior product}"'] & & \mathrm{BV} \quad \Delta \arrow[dd, "\text{inclusion}"] & & \mathrm{Lie} \arrow[ull, bend right, "\text{inclusion}"'] \arrow[dddll, bend left, "\text{Koszul bracket}"] \\
 & \underset{\text{hypercom. operad}}{H} \arrow[dr, "\simeq"] & & & \\
 & & \mathrm{BV}/\Delta \quad i_\pi \arrow[d] & & \\
 & & \mathrm{End}(\Omega(M)) & &
\end{tikzcd}
$$

A hypercommutative algebra A is trivial if it reduces to the underlying commutative algebra (A, m_2) i.e. $m_n = 0$ for $n > 2$

Operadically:

$$\mathcal{H} \longrightarrow \mathrm{Com} \longrightarrow \mathrm{End}(A)$$

$$m_2 \longmapsto \mu \text{ generator}$$

$$n > 2 \quad m_n \longmapsto 0$$

Main theorem:

The previous hypercommutative algebra structure on $\Omega(M)$ is quasi isomorphic to the trivial one.

One can similarly say that:

- A BV-algebra is trivial if $\Delta = 0$
- A Gerstenhaber algebra is trivial if its underlying Lie algebra is abelian

Corollary:

The previous BV and Gerstenhaber algebra structures on $\Omega(M)$ are quasi isomorphic to the trivial ones.

The operad maps

$$\mathrm{Lie} \xrightarrow{\text{incl.}} H \longrightarrow \mathrm{Com}$$

induce

$$\mathrm{Lie}_\infty \xrightarrow{\text{incl.}} H_\infty \longrightarrow \mathrm{Com}_\infty$$

so we also have a notion of trivial H_∞-algebra

Corollary:

The previous H_∞-algebra structure on $H^*(M)$ is ∞-isomorphic to the trivial one.

And similarly for BV_∞ and G_∞.

$$
\begin{tikzcd}
 & & \mathrm{Gerst} \arrow[d] & & \\
\mathrm{Comm} \arrow[urr, bend left, "\text{inclusion}"] \arrow[dr, "\text{inclusion}"] \arrow[dddddrr, bend right, "\text{Exterior product}"'] & & \mathrm{BV} \quad \Delta \arrow[dd, "\text{inclusion}"] & & \mathrm{Lie} \arrow[ull, bend right, "\text{inclusion}"'] \arrow[dddddll, bend left, "\text{Koszul bracket}"] \\
 & H \arrow[dr, "\simeq"] & & & \\
 & & \mathrm{BV}/\Delta \quad i_\pi \arrow[d] & & \\
 & & \mathrm{EBV} \arrow[d] & \text{exact Batalin Vilkovisky operad} & \\
 & & \mathrm{End}(\Omega(M)) & &
\end{tikzcd}
$$

Exact Batalin-Vilkovisky algebra

$\equiv$ commutative algebra

+ differential operator i of degree -2 and order ≤ 2

+ $[i,[i,d]] = 0$

Example: $\Omega(M)$ with $i = i_\pi$

$$\begin{array}{rccc} & BV/\Delta & \longrightarrow & EBV \\ & \phi_1 & \longmapsto & i \\ n \geq 2 & \phi_n & \longmapsto & 0 \end{array}$$

Theorem:

The operad map $\mathrm{Com} \longrightarrow \mathrm{EBV}$ is a quasi-isomorphism.

Technical theorem: Operad quasi-isomorphism machine

Free operad on a Σ_*-module C

Θ operad, $\mathcal{P} = \Theta \amalg \mathcal{F}(C)$ coproduct
weighted
weight 0
↳ weight 1

$\cup$

operadic ideal (S)
↳ weight homogeneous, positive weight

$C \circlearrowleft h$ contraction: $|h| = -1$, $dh + hd = 1$, $h^2 = 0$

$\mathcal{P} \circlearrowleft h$ extension such that $h(\Theta) = 0$

$$h(x \circ_i y) = \frac{w(x)}{w(x)+w(y)}\, h(x) \circ_i y + (-1)^{|x|} \frac{w(y)}{w(x)+w(y)}\, x \circ_i h(y)$$

If $h(S) \subset (S) \implies \Theta \xrightarrow[\text{incl.}]{\simeq} \mathcal{P}/(S)$ is a quasi-isomorphism

Example :

$$\mathrm{EBV} = \mathrm{Com} \amalg \mathcal{F}(C) \,/\, (S)$$ with

C is the complex

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & k \cdot i \arrow[r] & k \cdot d(i) \arrow[r] \arrow[l, bend right, "h"'] & 0 \arrow[r] & \cdots \end{tikzcd}$$

degree $\quad -2 \qquad -1$

concentrated in arity 1.

S has two elements: $\qquad h(1.) = h(2.) = 0 \in (S)$

1. $i\,d(i) - d(i)\,i$ $\quad$ weight 2
2. $(\mu^2 \circ_1 i) \cdot [\,() + (1\ 2) + (1\ 2\ 3)\,] + i\mu^2 + \mu \circ_1 (i\mu) \cdot [\,() + (2\ 3) + (1\ 3\ 2)\,]$ $\quad$ weight 1

Thanks for
your attention!

Question:

Any operad map $f: \mathcal{O} \longrightarrow \mathcal{Q}$ factors as

$$\mathcal{O} \underset{\text{trivial cofibration}}{\overset{\sim}{\rightarrowtail}} \mathcal{P} = \mathcal{O} \amalg \mathcal{F}_i(C) \underset{\text{fibration}}{\twoheadrightarrow} \mathcal{P}/(S) \cong \mathcal{Q}$$

$C \supseteq h$

Suppose f is an injective quasi-iso.

Can we always get h and S satisfying the theorem?