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The 1-type of a Waldhausen K-Theory Spectrum

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Abstract

Waldhausen's K-theory of a category $\mathbf{C}$ with cofibrations and weak equivalences (see [5, 6]) extends the classical notions of the K-theory of rings, additive categories and exact categories.

In this poster we present a new algebraic model $\mathcal{D}_*\mathbf{C}$ for the 1-type $P_1K\mathbf{C}$ of the Waldhausen K-theory spectrum $K\mathbf{C}$. The object $\mathcal{D}_*\mathbf{C}$ is a *stable quadratic module* [1], and consists of a diagram of groups

$$\begin{tikzcd} & (\mathcal{D}_0\mathbf{C})^{ab}\otimes(\mathcal{D}_0\mathbf{C})^{ab} \arrow[d, "\langle\cdot,\cdot\rangle"] & & \\ K_1\mathbf{C} \arrow[r] & \mathcal{D}_1\mathbf{C} \arrow[r, "\partial"] & \mathcal{D}_0\mathbf{C} \arrow[r] & K_0\mathbf{C} \end{tikzcd} \qquad \text{(A)}$$

in which the bottom row is exact.

Equivalently, $\mathcal{D}_*\mathbf{C}$ can be thought of as a strict symmetric monoidal category in which all the objects are strictly invertible with respect to the monoidal structure.

Our model $\mathcal{D}_*\mathbf{C}$ is defined by a presentation in terms of generators and relations in the spirit of Nenashev, who gave a model for K_1 of an exact category in [3]. The important features of $\mathcal{D}_*\mathbf{C}$ are the following:

- It is *small*, as it has generators given by the objects, the weak equivalences and the cofiber sequences of the category $\mathbf{C}$.
- It has *minimal nilpotency degree*, since the groups $\mathcal{D}_0\mathbf{C}$ and $\mathcal{D}_1\mathbf{C}$ have nilpotency class two.
- It encodes the 1-type in a *functorial* way, and there is a bijection between the homotopy classes of morphisms $P_1K\mathbf{C}\to P_1K\mathbf{D}$ and of morphisms $\mathcal{D}_*\mathbf{C}\to\mathcal{D}_*\mathbf{D}$.

Stable quadratic modules

A *stable quadratic module* C_* is a diagram of group homomorphisms

$$C_0{}^{ab}\otimes C_0{}^{ab}\xrightarrow{\langle -,-\rangle}C_1\xrightarrow{\partial}C_0$$

such that

$$\begin{aligned}\partial\langle c_0,d_0\rangle &= [d_0,c_0],\\ \langle\partial(c_1),\partial(d_1)\rangle &= [d_1,c_1],\\ \langle c_0,d_0\rangle+\langle d_0,c_0\rangle &= 0.\end{aligned}$$

Here $(-)^{ab}$ is abelianization and $[a,b]=-a-b+a+b$ is the commutator.

Let $C_0\ltimes C_1$ be the semidirect product group, where C_0 acts on C_1 by

$$c_1\cdot c_0 \;=\; c_1+\langle c_0,\partial c_1\rangle\,.$$

A stable quadratic module can be regarded as a monoidal category with object set C_0 and morphisms given by elements of $C_0\ltimes C_1$,

$$(c_0,c_1)\colon c_0\longrightarrow c_0+\partial(c_1)\,.$$

The group structures on C_0 and $C_0\ltimes C_1$ induce a symmetric monoidal structure with symmetry isomorphism given by the bracket $\langle -,-\rangle$,

$$(c_0+d_0,\langle c_0,d_0\rangle)\colon c_0+d_0\longrightarrow d_0+c_0.$$

The *classifying spectrum* of a symmetric monoidal category is defined by Segal in [4]. In the case of a stable quadratic module a simpler construction can be derived from [2].

For the proof of our theorem it is also important to note that a stable quadratic module C_* is a particular case of a commutative monoid in the category of *crossed complexes*. Indeed, we obtain $\mathcal{D}_*\mathbf{C}$ as the quotient of such a monoid $\mu\colon\Pi\otimes\Pi\to\Pi$ by the relations

$$\mu(a\otimes[b,c]) \;=\; 0 \qquad\text{and}\qquad d\;=\;0$$

for all $a,b,c\in\Pi_1$ and $d\in\Pi_{\geq 3}$.

Waldhausen's $S_\bullet$ construction

Let $\mathbf{C}$ be a Waldhausen category with zero object $*$. We denote the cofiber sequences and weak equivalences respectively by

$$A\rightarrowtail B\twoheadrightarrow B/A, \qquad\qquad A\xrightarrow{\sim}A'.$$

Waldhausen defined the K-theory of $\mathbf{C}$ as the homotopy of a certain simplicial category $wS_\bullet\mathbf{C}$ (or that of the diagonal of the bisimplicial set $X_{\bullet,\bullet}$ given by the nerve of this simplicial category.)

$$\boxed{\sigma_{2,1}}\qquad \begin{tikzcd} A' \arrow[r, tail] & B' \arrow[r, two heads] & B'/A' \\ A \arrow[r, tail] \arrow[u, "\sim"] & B \arrow[r, two heads] \arrow[u, "\sim"] & B/A \arrow[u, "\sim"] \end{tikzcd}$$

$$\boxed{\sigma_{3,0}}\qquad \begin{tikzcd} & & C/B \\ & B/A \arrow[r, tail] & C/A \arrow[u, two heads] \\ A \arrow[r, tail] & B \arrow[r, tail] \arrow[u, two heads] & C \arrow[u, two heads] \end{tikzcd}$$

Figure 1: Elements $\sigma_{2,1}$ and $\sigma_{3,0}$ in the nerve $X_{\bullet,\bullet}$ of $wS_\bullet\mathbf{C}$.

A small presentation for K_0 and K_1

Definition $\mathcal{D}_*\mathbf{C}$ is the stable quadratic module with generators

$$\begin{aligned} [A] &\in \mathcal{D}_0\mathbf{C} && \text{for any object in } \mathbf{C},\\ [A\xrightarrow{\sim}A'] &\in \mathcal{D}_1\mathbf{C} && \text{for any weak equivalence in } \mathbf{C},\\ [A\rightarrowtail B\twoheadrightarrow B/A] &\in \mathcal{D}_1\mathbf{C} && \text{for any cofiber sequence in } \mathbf{C},\end{aligned}$$

subject to following seven relations. First, the degeneracy and bounday relations:

$$[*] \;=\; 0, \qquad [A\underset{\sim}{\xrightarrow{1_A}}A] \;=\; [A\overset{1_A}{\rightarrowtail}A\twoheadrightarrow *] \;=\; [*\rightarrowtail A\overset{1_A}{\twoheadrightarrow}A] \;=\; 0, \tag{1}$$

$$\partial[A\xrightarrow{\sim}A'] \;=\; -[A']+[A], \tag{2}$$

$$\partial[A\rightarrowtail B\twoheadrightarrow B/A] \;=\; -[B]+[B/A]+[A]. \tag{3}$$

Also, for any composite of weak equivalences $A\xrightarrow{\sim}A'\xrightarrow{\sim}A''$,

$$[A\xrightarrow{\sim}A''] \;=\; [A'\xrightarrow{\sim}A'']+[A\xrightarrow{\sim}A']. \tag{4}$$

For any diagram $\sigma_{2,1}$ as in Figure 1,

$$\begin{aligned}[B\xrightarrow{\sim}B']+[A\rightarrowtail B\twoheadrightarrow B/A] \;=\; &[A'\rightarrowtail B'\twoheadrightarrow B'/A']+[A\xrightarrow{\sim}A']+[B/A\xrightarrow{\sim}B'/A']\\ &+\langle [A],-[B'/A']+[B/A]\rangle.\end{aligned} \tag{5}$$

For any diagram $\sigma_{3,0}$ as in Figure 1,

$$\begin{aligned}[B\rightarrowtail C\twoheadrightarrow C/B]+[A\rightarrowtail B\twoheadrightarrow B/A] \;=\; &[A\rightarrowtail C\twoheadrightarrow C/A]+[B/A\rightarrowtail C/A\twoheadrightarrow C/B]\\ &+\langle [A],-[C/A]+[C/B]+[B/A]\rangle.\end{aligned} \tag{6}$$

For any coproduct diagram $A\underset{p_1}{\overset{i_1}{\rightleftarrows}}A\vee B\underset{p_2}{\overset{i_2}{\leftrightarrows}}B$ in $\mathbf{C}$,

$$\langle [A],[B]\rangle \;=\; -[A\overset{i_1}{\rightarrowtail}A\vee B\overset{p_2}{\twoheadrightarrow}B]+[B\overset{i_2}{\rightarrowtail}A\vee B\overset{p_1}{\twoheadrightarrow}A]. \tag{7}$$

Main Theorem

Theorem B *The classifying spectrum of $\mathcal{D}_*\mathbf{C}$ is naturally isomorphic to the 1-type $P_1K\mathbf{C}$ of the Waldhausen K-theory spectrum of $\mathbf{C}$ in the stable homotopy category.*

The exact sequence in the row of (A) follows from this theorem. This theorem also allows us to describe the action of the stable Hopf map $\eta\in\pi_1S^0$ on $K_0\mathbf{C}$ in terms of generators and relations. For any object A in $\mathbf{C}$ there is an interchange map

$$\tau_A\colon A\vee A\cong A\vee A.$$

Moreover, by relation (2) we have $\partial[\tau_A]=0$ so we can regard $[\tau_A]$ as an element in $K_1\mathbf{C}$.

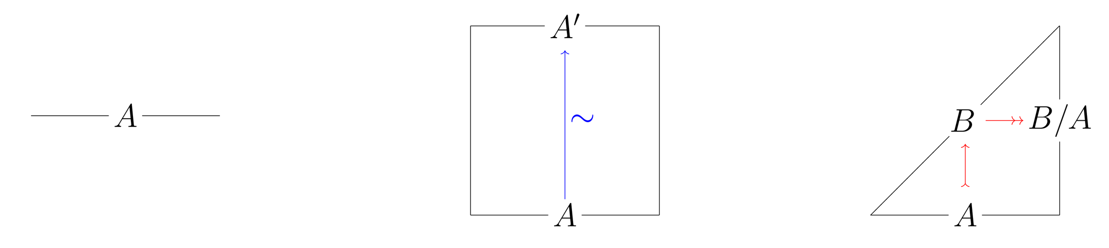
Corollary C *The homomorphism $\cdot\eta\colon K_0\mathbf{C}\otimes\mathbb{Z}/2\to K_1\mathbf{C}$ induced by the action of the stable Hopf map $\eta\in\pi_1S^0=\mathbb{Z}/2$ is given by*

$$[A]\cdot\eta \;=\; [A\vee A\underset{\sim}{\xrightarrow{\tau_A}}A\vee A].$$

Informal illustrations

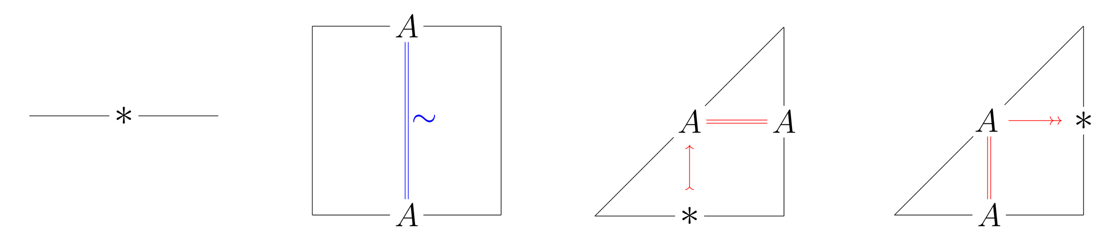
Some understanding of the presentation of the 1-type can be gained by imagining the cells of the bisimplicial set $X_{\bullet,\bullet}$ as building blocks to be assembled, then flattened in dimensions above two. Thus:

Generators in $\mathcal{D}_0\mathbf{C}$ and $\mathcal{D}_1\mathbf{C}$ are the 1- and 2-cells of $X_{\bullet,\bullet}$:

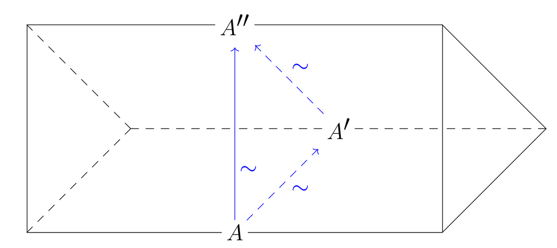


Relations (2) and (3) express the boundaries of these 2-cells.

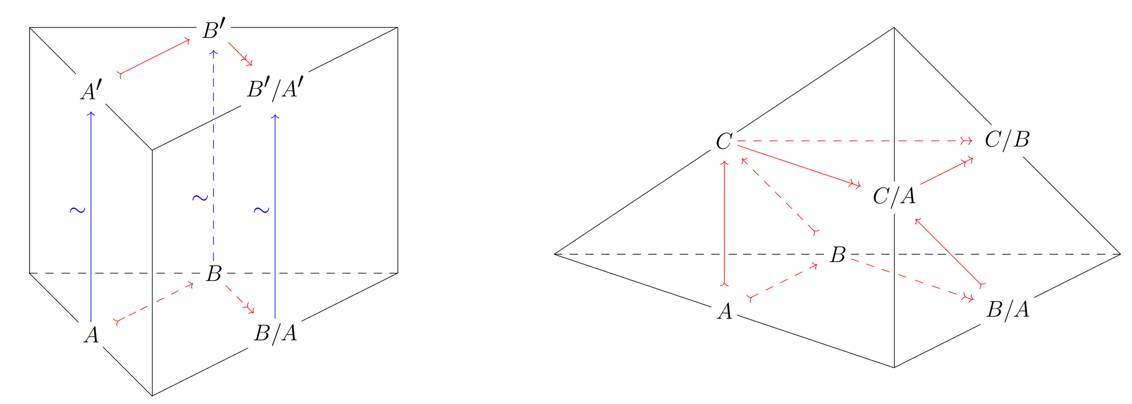
Relation (1), on the other hand, says that the degenerate cells are trivial:



Relation (4) says the following 3-cell is a *commutative prism*, in the sense that the front face is equal to the sum of the back two faces.



Two more types of commutative 3-cell exist, corresponding to elements $\sigma_{2,1}$ and $\sigma_{3,0}$ in Figure 1. These express relations (5) and (6).



Relation (7) arises from the multiplicative structure

$$\Pi(X_{\bullet,\bullet})\otimes\Pi(X_{\bullet,\bullet})\xrightarrow{\mu}\Pi(X_{\bullet,\bullet})$$

on the *total crossed complex* of $X_{\bullet,\bullet}$. This is induced, via the shuffle map, from the monoid structure on $X_{\bullet,\bullet}$ given by the coproduct on $\mathbf{C}$.

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