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The symmetric action on secondary homotopy groups

Hans-Joachim Baues Fernando Muro*

Abstract

We show that the symmetric track groups $\mathrm{Sym}_\square(n)$, which are extensions of the symmetric groups $\mathrm{Sym}(n)$ associated to the second Stiefel-Whitney class, act as crossed modules on the secondary homotopy groups of a pointed space.

Introduction

Secondary homotopy operations like Toda brackets [Tod62] or cup-one products [BJM83], [HM93], are defined by pasting tracks, where tracks are homotopy classes of homotopies. Since secondary homotopy operations play a crucial role in homotopy theory it is of importance to develop the algebraic theory of tracks. We do this by introducing secondary homotopy groups of a pointed space X

$$\Pi_{n,*}X = \left(\Pi_{n,1}X \xrightarrow{\partial} \Pi_{n,0}X\right)$$

which have the structure of a quadratic pair module, see Section 1. Here ∂ is a group homomorphism with cokernel $\pi_n X$ and kernel $\pi_{n+1}X$ for $n \geq 3$.

We define $\Pi_{n,*}X$ for $n \geq 2$ directly in terms of maps $S^n \to X$ and tracks from such maps to the trivial map. For $n \geq 0$ the functor $\Pi_{n,*}$ is an additive version of the

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functor $\pi_{n,*}$ studied in [BM08]. The homotopy category of $(n-1)$-connected $(n+1)$-types is equivalent via $\Pi_{n,*}$ to the homotopy category of quadratic pair modules for $n \geq 3$.

In this paper we consider the "generalized coefficients" of secondary homotopy groups $\Pi_{n,*}X$ obtained by the action of the symmetric group $\mathrm{Sym}(n)$ on $S^n = S^1 \wedge \overset{n}{\cdots} \wedge S^1$ via permutation of coordinates. For a permutation $\sigma \in \mathrm{Sym}(n)$ the map $\sigma\colon S^n \to S^n$ has degree $\operatorname{sign}\sigma \in \{\pm 1\}$. The group $\{\pm 1\}$ also acts on S^n by using the topological abelian group structure of S^1 and suspending $n-1$ times. This shows that there are tracks $\sigma \Rightarrow \operatorname{sign}\sigma$ which, by definition, are the elements of the symmetric track group $\mathrm{Sym}_\square(n)$. Also these tracks act on $\Pi_{n,*}X$. We clarify this action by showing that the group $\mathrm{Sym}_\square(n)$ gives rise to a crossed module which acts as a crossed module on the quadratic pair module $\Pi_{n,*}X$.

The symmetric track group is a central extension

$$\mathbb{Z}/2 \hookrightarrow \mathrm{Sym}_\square(n) \overset{\delta}{\twoheadrightarrow} \mathrm{Sym}(n)$$

which, as we show, represents the second Stiefel-Whitney class pulled back to $\mathrm{Sym}(n)$. The symmetric track group is computed in Section 6. We actually compute a faithful positive pin representation of $\mathrm{Sym}_\square(n)$ from which we derive a finite presentation of this group. This group also arose in a different way in the work of Schur [Sch11] and Serre [Ser84].

In [BM07] we describe the smash product operation on secondary homotopy groups $\Pi_{n,*}X$. This operation endows $\Pi_{*,*}$ with the structure of a lax symmetric monoidal functor where the crossed module action of $\mathrm{Sym}_\square(n)$ on $\Pi_{n,*}X$ is of crucial importance. This leads to an algebraic approximation of the symmetric monoidal category of spectra by secondary homotopy groups, see [BM06]. As an example we prove a formula for the unstable cup-one product $\alpha \smile_1 \alpha \in \pi_{2n+1}S^{2m}$ of an element $\alpha \in \pi_n S^m$ where n and m are even. We show that

$$2(\alpha \smile_1 \alpha) = \frac{n+m}{2}(\alpha \wedge \alpha)(\Sigma^{2(n-1)}\eta)$$

where $\eta\colon S^3 \to S^2$ is the Hopf map. In the very special case when $n/2$ is odd and $m/2$ is even then this formula was achieved by totally different methods in [BJM83].

1 Square groups and quadratic pair modules

In this section we describe the algebraic concepts needed for the structure of secondary homotopy groups.

Definition 1.1. A *square group* X is a diagram

$$X = (X_e \underset{H}{\overset{P}{\leftrightarrows}} X_{ee})$$

where X_e is a group with an additively written group law, X_{ee} is an abelian group, P is a homomorphism, H is a function such that the cross effect

$$(a|b)_H = H(a+b) - H(b) - H(a)$$

is linear in a and $b \in X_e$, and the following relations are satisfied for all $x, y \in X_{ee}$,

1. $(Px|b)_H = 0$, $(a|Py) = 0$,
2. $P(a|b)_H = -a - b + a + b$,
3. $PHP(x) = P(x) + P(x)$.

These relations imply that the image of P is central in X_e, and that X_e is a group of nilpotency class 2.

The function

$$T = HP - 1 \colon X_{ee} \longrightarrow X_{ee}$$

is an involution, i. e. a homomorphism with $T^2 = 1$.

A morphism of square groups $f \colon X \to Y$ is given by homomorphisms

$$f_e \colon X_e \longrightarrow Y_e,$$

$$f_{ee} \colon X_{ee} \longrightarrow Y_{ee},$$

commuting with P and H.

Let **SG** be the category of square groups. A square group X with $X_{ee} = 0$ is the same as an abelian group X_e. This yields the full inclusion of categories $\mathbf{Ab} \subset \mathbf{SG}$ where **Ab** is the category of abelian groups.

Square groups were introduced in [BP99] to describe quadratic endofunctors of the category **Gr** of groups. More precisely, any square group X gives rise to a quadratic functor

$$- \otimes X \colon \mathbf{Gr} \longrightarrow \mathbf{Gr}.$$

Given a group G the group $G \otimes X$ is generated by the symbols $g \otimes x$ and $[g,h] \otimes z$, $g, h \in G$, $x \in X_e$, $z \in X_{ee}$ subject to the relations

$$\begin{aligned} (g+h) \otimes x &= g \otimes x + h \otimes x + [g,h] \otimes H(x), \\ [g,g] \otimes z &= g \otimes P(z), \end{aligned}$$

where $g \otimes x$ is linear in x and $[g,h] \otimes z$ is central and linear in each variable g, h, z. If X is an abelian group then $G \otimes X = G_{\mathrm{ab}} \otimes X_e$, where G_{ab} is the abelianization of a group G. In fact, any quadratic functor $F \colon \mathbf{Gr} \to \mathbf{Gr}$ which preserves reflexive coequalizers and filtered colimits has the form $F = - \otimes X$, see [BP99]. The theory of square groups is discussed in detail in [BJP05].

There is a natural isomorphism

$$X_e \xrightarrow{\cong} \mathbb{Z} \otimes X, \quad x \mapsto 1 \otimes x.$$

In particular the homomorphism $n \colon \mathbb{Z} \to \mathbb{Z}$ induces a homomorphism $n^* \colon X_e \to X_e$ fitting into the following commutative diagram

$$\begin{array}{ccc} \mathbb{Z} \otimes X & \xrightarrow{n \otimes X} & \mathbb{Z} \otimes X \\ \cong \uparrow & & \uparrow \cong \\ X_e & \xrightarrow{n^*} & X_e \end{array} \qquad (1.2)$$

The homomorphism n^* is explicitly given by the following formula,

$$n^*x = n\cdot x + \binom{n}{2} PH(x).$$

Here we set $\binom{n}{2} = \frac{n(n-1)}{2}$ and for any additively written group G and any $n \in \mathbb{Z}$, $g \in G$,

$$n\cdot g = \begin{cases} g + \overset{n}{\cdots} + g, & \text{if } n \geq 0; \\ \\ -g - \overset{-n}{\cdots} - g, & \text{if } n < 0. \end{cases}$$

The function $n\cdot : G \to G$ in general is not a homomorphism, but if G is abelian then $n\cdot$ is a homomorphism. This homomorphism is generalized by n^* in (1.2) for square groups.

Definition 1.3. A *quadratic pair module* C is a morphism $\partial\colon C_{(1)} \to C_{(0)}$ between square groups

$$\begin{aligned} C_{(0)} &= (C_0 \underset{H}{\overset{P_0}{\leftrightarrows}} C_{ee}), \\ C_{(1)} &= (C_1 \underset{H_1}{\overset{P}{\leftrightarrows}} C_{ee}), \end{aligned}$$

such that $\partial_{ee} = 1\colon C_{ee} \to C_{ee}$ is the identity homomorphism. In particular ∂ is completely determined by the diagram

$$\begin{array}{ccc} & C_{ee} & \\ {\scriptstyle P}\swarrow & & \nwarrow{\scriptstyle H} \\ C_1 & \xrightarrow[\partial]{} & C_0 \end{array} \tag{1.4}$$

where $\partial = \partial_e$, $H_1 = H\partial$ and $P_0 = \partial P$.

The *homology* of a quadratic pair module C is given by the abelian groups

$$\begin{aligned} h_0 C &= C_0/\partial(C_1), \\ h_1 C &= \mathrm{Ker}[\partial\colon C_1 \to C_0]. \end{aligned} \tag{1.5}$$

Morphisms of quadratic pair modules $f\colon C \to D$ are given by group homomorphisms $f_0\colon C_0 \to D_0$, $f_1\colon C_1 \to D_1$, $f_{ee}\colon C_{ee} \to D_{ee}$, commuting with H, P and ∂ in (1.4) as in the diagram

$$\begin{array}{ccccccc} C_0 & \xrightarrow{H} & C_{ee} & \xrightarrow{P} & C_1 & \xrightarrow{\partial} & C_0 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{ee}} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ D_0 & \xrightarrow{H} & D_{ee} & \xrightarrow{P} & D_1 & \xrightarrow{\partial} & D_0 \end{array}$$

They form a category denoted by **qpm**. A morphism in **qpm** is said to be a *weak equivalence* if it induces isomorphisms in h_0 and h_1.

Quadratic pair modules are also the objects of a bigger category **wqpm** given by weak morphisms. A *weak morphism* $f\colon C \to D$ between quadratic pair modules is given by three homomorphisms f_0, f_1, f_{ee} as above, but we only require the following two diagrams to be commutative

$$\begin{array}{ccc} C_{ee} & \xrightarrow{T} & C_{ee} \\ \downarrow{\scriptstyle f_{ee}} & & \downarrow{\scriptstyle f_{ee}} \\ D_{ee} & \xrightarrow{T} & D_{ee} \end{array} \qquad \begin{array}{ccccccc} \otimes^2(C_0)_{\mathrm{ab}} & \xrightarrow{(-|-)_H} & C_{ee} & \xrightarrow{P} & C_1 & \xrightarrow{\partial} & C_0 \\ \downarrow{\scriptstyle \otimes^2(f_0)_{\mathrm{ab}}} & & \downarrow{\scriptstyle f_{ee}} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ \otimes^2(D_0)_{\mathrm{ab}} & \xrightarrow{(-|-)_H} & D_{ee} & \xrightarrow{P} & D_1 & \xrightarrow{\partial} & D_0 \end{array}$$

Here $\otimes^2 A = A \otimes A$ denotes the tensor square of an abelian group. Therefore **qpm** $\subset$ **wqpm** is a subcategory with the same objects.

Let $(\mathbb{Z}, \cdot)$ be the multiplicative (abelian) monoid of the integers $\mathbb{Z}$.

Definition 1.6. Any quadratic pair module C admits an action of $(\mathbb{Z}, \cdot)$ given by the morphisms $n^* \colon C \to C$ in **wqpm**, $n \in \mathbb{Z}$, defined by the equations

- $n^*x = n \cdot x + \binom{n}{2}\partial PH(x)$ for $x \in C_0$,
- $n^*y = n \cdot y + \binom{n}{2}PH\partial(y)$ for $y \in C_1$,
- $n^*z = n^2 z$ for $z \in C_{ee}$.

We point out that $n^* \colon C \to C$ is an example of a weak morphism which is not a morphism in **qpm** since n^* is not compatible with H. Notice that $n^* \colon C_0 \to C_0$ and $n^* \colon C_1 \to C_1$ are induced by the square group morphisms $n \otimes C_{(0)}$ and $n \otimes C_{(1)}$ respectively, see diagram (1.2). We emphasize that this action is always defined for any quadratic pair module C and it is natural in the following sense, for any morphism $f \colon C \to D$ in **qpm** and any $n \in \mathbb{Z}$, the equality

$$fn^* = n^*f$$

holds. This property does not hold if f is a weak morphism. The existence of this action should be compared to the fact that abelian groups are $\mathbb{Z}$-modules.

The category **squad** of stable quadratic modules is described in [Bau91, IV.C] and [BM08]. Quadratic modules in general are discussed in [Bau91] and [Ell93], they are special 2-crossed modules in the sense of [Con84]. More precisely, a *stable quadratic module* C is a diagram of group homomorphisms

$$\otimes^2(C_0)_{\mathrm{ab}} \xrightarrow{\omega} C_1 \xrightarrow{\partial} C_0,$$

such that given $c_i, d_i \in C_i$, $i = 0, 1$,

$$\begin{aligned} \partial\omega(c_0 \otimes d_0) &= -c_0 - d_0 + c_0 + d_0, \\ \omega(\partial(c_1) \otimes \partial(d_1)) &= -c_1 - d_1 + c_1 + d_1, \\ \omega(c_0 \otimes d_0 + d_0 \otimes c_0) &= 0. \end{aligned}$$

Morphisms $f \colon C \to D$ in **squad** are given by homomorphisms $f_i \colon C_i \to D_i$, $i = 0, 1$, compatible with ω and ∂. There is a forgetful functor from quadratic pair modules and weak morphisms to stable quadratic modules

$$\mathbf{wqpm} \longrightarrow \mathbf{squad} \tag{1.7}$$

sending C as in Definition 1.3 to the stable quadratic module

$$\otimes^2(C_0)_{\mathrm{ab}} \xrightarrow{P(-|-)_H} C_1 \xrightarrow{\partial} C_0. \tag{1.8}$$

This functor is faithful over the full subcategory of quadratic pair modules such that the cross effect of H is an isomorphism $(-|-)_H \colon \otimes^2 (C_0)_{\mathrm{ab}} \cong C_{ee}$.

A *track category* is a groupoid-enriched category, which is also a 2-category where all 2-morphisms (also termed *tracks*) are vertically invertible. The vertical composition in track categories is denoted by $\Box$, the vertical inverse of a track α is $\alpha^{\boxminus}$, and the trivial track from a morphism f to itself is $0^{\Box} \colon f \Rightarrow f$.

Remark 1.9. The category $\mathbf{Top}^*$ of pointed spaces is known to be a track category with tracks given by homotopy classes of homotopies. This track category has in addition a *strict zero object* $*$, which is an object such that the morphism groupoids from or to $*$ are trivial, i.e. they consist of only one object and one morphism. In particular the zero morphism $0 \colon X \to Y$ between two objects X, Y is uniquely defined as the morphism which factor as $X \to * \to Y$. Moreover, in a track category with a strict zero object the following crucial fact holds: The horizontal composition of any track H and a zero morphism is a trivial track $H0 = 0^{\Box}$, $0H = 0^{\Box}$.

The forgetful functor (1.7) can be used to pull-back to **wqpm** the track category structure on **squad** introduced in [BM08, 6]. The track structure on **squad** was already a pull-back along the forgetful functor

$$\mathbf{squad} \longrightarrow \mathbf{cross} \tag{1.10}$$

from stable quadratic modules to crossed modules considered also in [BM08, 6].

Definition 1.11. We recall that a *crossed module* $\partial \colon M \to N$ is a group homomorphism such that N acts on the right of M (the action will be denoted exponentially) and the homomorphism ∂ satisfies the following two properties ($m, m' \in M, n \in N$):

1. $\partial(m^n) = -n + \partial(m) + n$,
2. $m^{\partial(m')} = -m' + m + m'$.

The crossed module associated via (1.7) and (1.10) to a quadratic pair module C is given by the homomorphism

$$\partial \colon C_1 \longrightarrow C_0,$$

where C_0 acts on the right of C_1 by the formula, $x \in C_1$, $y \in C_0$,

$$x^y = x + P(\partial(x)|y)_H. \tag{1.12}$$

Definition 1.13. A *track* $\alpha \colon f \Rightarrow g$ between two morphisms $f, g \colon C \to D$ in **wqpm** is a function

$$\alpha \colon C_0 \longrightarrow D_1$$

satisfying the equations, $x, y \in C_0$, $z \in C_1$,

1. $\alpha(x + y) = \alpha(x)^{f_0(y)} + \alpha(y)$,

2. $g_0(x) = f_0(x) + \partial\alpha(x)$,
3. $g_1(z) = f_1(z) + \alpha\partial(z)$.

Tracks in **qpm** are tracks in **wqpm** between morphisms in the subcategory **qpm** $\subset$ **wqpm**.

Proposition 1.14. *The categories* **wqpm** *and* **qpm** *are track categories with the tracks in Definition 1.13.*

This proposition is a direct consequence of [BM08, 6.4]. Vertical and horizontal compositions are defined in the proof of [BM08, 6.4].

The following result shows that the weak action of $(\mathbb{Z}, \cdot)$ defined above is also natural with respect to tracks in **qpm**.

Proposition 1.15. *Let $f, g\colon C \to D$ be morphisms in* **qpm** *and let $\alpha\colon g \Rightarrow f$ be a track as in Definition 1.13. Then the following diagram commutes*

$$\begin{array}{ccc} C_0 & \xrightarrow{\alpha} & D_1 \\ {\scriptstyle n^*}\downarrow & & \downarrow{\scriptstyle n^*} \\ C_0 & \xrightarrow{\alpha} & D_1 \end{array}$$

Given a pointed set E with base point $* \in E$ we denote by $\langle E\rangle_{\mathrm{nil}}$ and $\mathbb{Z}[E]$ the free group of nilpotency class 2 and the free abelian group generated by E with $* = 0$ respectively. More generally G_{nil} denotes the projection of a group G to the variety of groups of nilpotency class 2.

Definition 1.16. A quadratic pair module C is said to be 0-*free* if $C_0 = \langle E\rangle_{\mathrm{nil}}$, $C_{ee} = \otimes^2\mathbb{Z}[E]$ and H is determined by the equalities $H(e) = 0$ for any $e \in E$ and $(s|t)_H = t \otimes s$ for any $s, t \in \langle E\rangle_{\mathrm{nil}}$. Notice that in this case the cross effect yields an isomorphism $(-|-)_H\colon \otimes^2 (C_0)_{\mathrm{ab}} \cong C_{ee}$.

One can similarly define a 0-free stable quadratic module as a stable quadratic module whose lower-dimensional group is free of nilpotency class 2. No further conditions are required in this case.

The next lemma shows that 0-free stable quadratic modules are in the image of the forgetful functor in (1.7).

Lemma 1.17. *Any* 0*-free stable quadratic module*

$$\otimes^2\mathbb{Z}[E] \xrightarrow{\omega} M \xrightarrow{\partial} \langle E\rangle_{\mathrm{nil}}$$

gives rise to a 0*-free quadratic pair module*

$$\begin{array}{ccc} & \otimes^2\mathbb{Z}[E] & \\ {\scriptstyle P}\swarrow & & \nwarrow{\scriptstyle H} \\ M & \xrightarrow[\partial]{} & \langle E\rangle_{\mathrm{nil}} \end{array}$$

with $P(a \otimes b) = \omega(b \otimes a)$.

Later we will need the following technical lemma which measures the lack of compatibility of certain tracks in **wqpm** with the action of $(\mathbb{Z},\cdot)$.

Lemma 1.18. *Let C be a 0-free quadratic pair module with $C_0 = \langle E\rangle_{\mathrm{nil}}$, let $f\colon C_0 \to C_0$ be an endomorphism induced by a pointed map $E \to E$, and let $\alpha\colon C_0 \to C_1$ be a map satisfying*

$$\begin{aligned}\alpha(x+y) &= \alpha(x)^{f(y)} + \alpha(y),\\ m^*x &= f(x) + \partial\alpha(x),\end{aligned}$$

for some $m \in \mathbb{Z}$ and any $x, y \in C_0$. Then the following formula holds for any $n \in \mathbb{Z}$ and $x \in C_0$.

$$\alpha(n^*x) = n^*\alpha(x) + \binom{m}{2}\binom{n}{2}P(x|x)_H.$$

Proof. We first check that the lemma holds for $x+y$ provided it holds for $x, y \in C_0$.

$$\begin{aligned}\alpha n^*(x+y) &= \alpha(n^*x + n^*y)\\ &= \alpha(n^*x)^{f(n^*y)} + \alpha(n^*y)\\ &= n^*\alpha(x) + n^*\alpha(y) + \binom{m}{2}\binom{n}{2}P(x|x)_H\\ &\quad + \binom{m}{2}\binom{n}{2}P(y|y)_H + P(-f(n^*x) + n^*m^*x|f(n^*y))_H\\ &= n^*(\alpha(x)+\alpha(y)) + n^*P(-f(x)+m^*x|fy)_H\\ &\quad + \binom{m}{2}\binom{n}{2}P(x+y|x+y)_H\\ &= n^*(\alpha(x)^{f(y)} + \alpha(y)) + \binom{m}{2}\binom{n}{2}P(x+y|x+y)_H\\ &= n^*\alpha(x+y) + \binom{m}{2}\binom{n}{2}P(x+y|x+y)_H.\end{aligned}$$

Here we use that f is compatible with the action of $(\mathbb{Z},\cdot)$ and that $P(x|x)_H$ is linear in x.

We now check that the lemma holds for $-x$ provided it holds for x. For this we use that, by the first equation of the statement, $\alpha(-y) = -\alpha(y)^{-f(y)}$.

$$\begin{aligned}\alpha n^*(-x) &= \alpha(-n^*x)\\ &= -(\alpha n^*x)^{-fn^*x}\\ &= -n^*\alpha(x) - \binom{m}{2}\binom{n}{2}P(x|x)_H - P(-n^*f(x) + n^*m^*x| - fn^*x)_H\\ &= -n^*\alpha(x) - \binom{m}{2}\binom{n}{2}P(x|x)_H - n^*P(-f(x)+m^*x| - fx)_H\\ &= -n^*(\alpha(x) + P(-f(x)+m^*x| - f(x))_H) + \binom{m}{2}\binom{n}{2}P(-x| - x)_H\\ &= -n^*\left(\alpha(x)^{-f(x)}\right) + \binom{m}{2}\binom{n}{2}P(-x| - x)_H\\ &= n^*\alpha(-x) + \binom{m}{2}\binom{n}{2}P(-x| - x)_H.\end{aligned}$$

Now since $C_0 = \langle E\rangle_{\mathrm{nil}}$ we only need to check that the proposition holds for $e \in E$. But $H(e) = 0$, so we have $n^*e = n\cdot e$. The equality

$$\alpha(n\cdot e) \;=\; n\cdot\alpha(e) + \binom{n}{2}P(f(e)|f(e))_H + m\binom{n}{2}P(e|f(e))_H$$

follows easily by induction in n from the first equation of the statement and the laws of a quadratic pair module. On the other hand

$$n^*\alpha(e) \;=\; n\cdot\alpha(e) + \binom{n}{2}PH(-f(e)+m\cdot e).$$

One can also check by induction that

$$PH(-f(e)+m\cdot e) \;=\; P(f(e)|f(e))_H + mP(e|f(e))_H - \binom{m}{2}P(e|e)_H.$$

Now the proof is finished. ■

Lemma 1.18 holds under the more general condition that C_0 is generated by elements $x \in C_0$ with $H(x) = 0$ and $Hf(x) = 0$.

2 Homotopy groups and secondary homotopy groups

Let $\mathbf{Top}^*$ be the category of (compactly generated) pointed spaces. Using classical homotopy groups $\pi_n X$ we obtain for $n \geq 0$ the functor

$$\Pi_n\colon \mathbf{Top}^* \longrightarrow \mathbf{Ab}$$

with

$$\Pi_n X = \begin{cases} \pi_n X, & n\geq 2,\\ (\pi_1 X)_{\mathrm{ab}}, & n=1,\\ \mathbb{Z}[\pi_0 X], & n=0, \end{cases} \tag{2.1}$$

termed *additive homotopy groups.*

One readily checks that the smash product

$$f\wedge g\colon S^n\wedge S^m \longrightarrow X\wedge Y$$

of maps $\{f\colon S^n\to X\}\in\pi_n X$ and $\{g\colon S^m\to Y\}\in\pi_m Y$ induces a well-defined homomorphism

$$\wedge\colon \Pi_n X\otimes\Pi_m Y \longrightarrow \Pi_{n+m}(X\wedge Y). \tag{2.2}$$

This homomorphism is symmetric in the sense that the interchange map $\tau_{X,Y}\colon X\wedge Y\to Y\wedge X$ yields the equation in $\Pi_{n+m}(Y\wedge X)$

$$(\tau_{X,Y})_*(f\wedge g) = (-1)^{nm} g\wedge f. \tag{2.3}$$

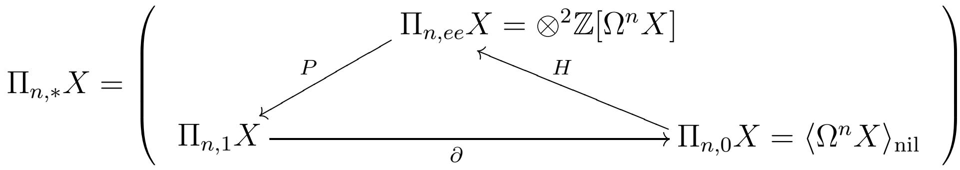
Here the sign $(-1)^{nm}$ is given by the interchange map

$$\tau_{n,m} = \tau_{S^n,S^m}\colon S^{n+m}\longrightarrow S^{m+n} \tag{2.4}$$

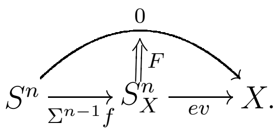
which has degree $(-1)^{nm}$. Here $\tau_{n,m}$ also designates the corresponding element of the symmetric group $\mathrm{Sym}(n+m)$ which acts from the left on S^{n+m}, see Section 5 below.

We want to generalize the smash product operator (2.2) for additive secondary homotopy groups.

Definition 2.5. Let $n \geq 2$. For a pointed space X we define the *additive secondary homotopy group* $\Pi_{n,*}X$ which is the 0-free quadratic pair module given by the diagram



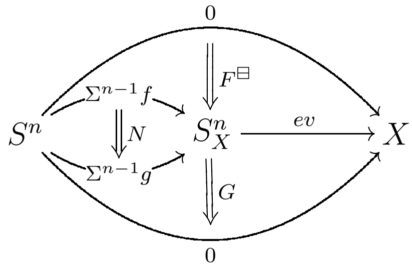
We obtain the group $\Pi_{n,1}X$ and the homomorphisms P and ∂ as follows. The group $\Pi_{n,1}X$ is given by the set of equivalence classes $[f, F]$ represented by a map $f\colon S^1 \to \vee_{\Omega^n X} S^1$ and a track



Here the pointed space

$$S^n_X = \vee_{\Omega^n X} S^n = \Sigma^n \Omega^n X$$

is the n-fold suspension of the n-fold loop space $\Omega^n X$, where $\Omega^n X$ is regarded in this equation as a pointed set with the discrete topology. Hence S^n_X is the coproduct of n-spheres indexed by the set of non-trivial maps $S^n \to X$, and $ev\colon S^n_X \to X$ is the obvious evaluation map. Moreover, for the sake of simplicity given a map $f\colon S^1 \to \vee_{\Omega^n X} S^1$ we will denote $f_{ev} = ev(\Sigma^{n-1}f)$, so that F in the previous diagram is a track $F\colon f_{ev} \Rightarrow 0$. The equivalence relation $[f, F] = [g, G]$ holds provided there is a track $N\colon \Sigma^{n-1}f \Rightarrow \Sigma^{n-1}g$ with $\mathit{Hopf}(N) = 0$ if $n \geq 3$ or $\bar{\sigma}\mathit{Hopf}(N) = 0$ if $n = 2$, see (2.6) and (2.7) below, such that the composite track in the following diagram is the trivial track.



That is $F = G\square(ev\, N)$. The map ∂ is defined by the formula

$$\partial[f, F] = (\pi_1 f)_{\mathrm{nil}}(1),$$

where $1 \in \pi_1 S^1 = \mathbb{Z}$.

The Hopf invariant of a track $N\colon \Sigma^{n-1}f \Rightarrow \Sigma^{n-1}g$ as above is defined in [BM08, 3.3] by the homomorphism

$$H_2(IS^1, S^1 \vee S^1) \stackrel{ad(N)_*}{\longrightarrow} H_2(\Omega^{n-1}S^n_X, \vee_{\Omega^n X}S^1) \cong \begin{cases} \hat{\otimes}^2\mathbb{Z}[\Omega^n X], & n \geq 3, \\ \otimes^2\mathbb{Z}[\Omega^n X], & n = 2, \end{cases} \tag{2.6}$$

which carries the generator $1 \in \mathbb{Z} \cong H_2(IS^1, S^1 \vee S^1)$ to $Hopf(N)$. Here $ad(N)_*$ is the homomorphism induced in homology by the adjoint of the homotopy

$$N\colon \Sigma^{n-1} IS^1 \cong IS^n \to S^n_X.$$

The reduced tensor square is given by

$$\hat{\otimes}^2 A = \frac{A \otimes A}{a \otimes b + b \otimes a \sim 0},$$

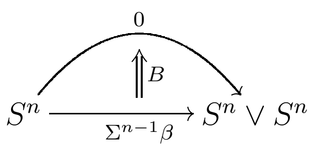
and

$$\bar{\sigma}\colon \otimes^2 A \twoheadrightarrow \hat{\otimes}^2 A, \quad \bar{\sigma}(a \otimes b) = a \hat{\otimes} b, \tag{2.7}$$

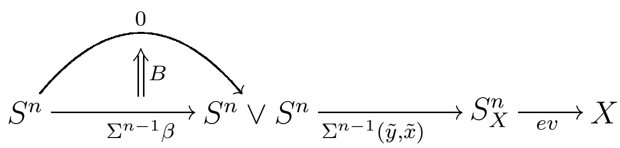
is the natural projection. The isomorphism in (2.6) is induced by the Pontrjagin product. We refer the reader to [BM08, 3] for a complete definition of the Hopf invariant for tracks and for the elementary properties which will be used in this paper. For the sake of simplicity we define the *reduced Hopf invariant* as $\overline{Hopf} = Hopf$ if $n \geq 3$ and $\overline{Hopf} = \bar{\sigma} Hopf$ if $n = 2$. A *nil-track* in this paper will be a track in $\mathbf{Top}^*$ with trivial reduced Hopf invariant. In particular the equivalence relation defining elements in $\Pi_{n,1}X$ is determined by nil-tracks.

This completes the definition of $\Pi_{n,1}X$, $n \geq 2$, as a set. The group structure of $\Pi_{n,1}X$ is induced by the comultiplication $\mu\colon S^1 \to S^1 \vee S^1$, compare [BM08, 4.4].

We now define the homomorphism P for additive secondary homotopy groups $\Pi_{n,*}X$ with $n \geq 2$. Consider the diagram



where $\beta\colon S^1 \to S^1 \vee S^1$ is given such that $(\pi_1\beta)_{\mathrm{nil}}(1) = -a-b+a+b \in \pi_1(S^1 \vee S^1)_{\mathrm{nil}}$ is the commutator of the generators a and b, which correspond to the inclusions of the first and the second factor of $S^1 \vee S^1$, respectively. The track B is any track with $\overline{Hopf}(B) = -a\hat{\otimes}b \in \hat{\otimes}^2 \pi_1(S^1 \vee S^1)_{\mathrm{ab}}$. Given $x \otimes y \in \otimes^2 \mathbb{Z}[\Omega^n X]$ let $\tilde{x}, \tilde{y}\colon S^1 \to \vee_{\Omega^n X} S^1$ be maps with $(\pi_1\tilde{x})_{\mathrm{ab}}(1) = x$ and $(\pi_1\tilde{y})_{\mathrm{ab}}(1) = y$. Then the diagram



(2.8)

represents an element

$$P(x \otimes y) = [(\tilde{y}, \tilde{x})\beta, ev\,(\Sigma^{n-1}(\tilde{y}, \tilde{x}))B] \in \Pi_{n,1}X.$$

This completes the definition of the quadratic pair module $\Pi_{n,*}X$ for $n \geq 2$. For $n = 0, 1$ we define the additive secondary homotopy groups $\Pi_{n,*}X$ by the following remark. In this way we get for $n \geq 0$ a functor

$$\Pi_{n,*}\colon \mathbf{Top}^* \longrightarrow \mathbf{qpm}$$

which is actually a track functor, see Remark 2.10 below.

We show in [BM08, 5.1] that, for $n \geq 3$, the homology of the additive secondary homotopy groups, in the sense of (1.5), is given by the classical homotopy groups

$$h_i \Pi_{n,*} X \;\cong\; \pi_{i+n} X, \;\; i = 0, 1.$$

For this we use that, in the range $n \geq 3$, the additive secondary homotopy groups considered here coincide with the (non-additive) secondary homotopy groups $\pi_{n,*}X$ defined in [BM08]. Moreover, by [BM08, 4.16] the n^{th} additive secondary homotopy group $\Pi_{n,*}(\vee_E S^n)$ of a wedge of n-spheres indexed by a pointed set E, $\vee_E S^n = \Sigma^n E$, is weakly equivalent to the 0-free quadratic pair module $\overline{\mathbb{Z}}_{\mathrm{nil}}[E]$ given by

$$\overline{\mathbb{Z}}_{\mathrm{nil}}[E] = \left(\begin{array}{ccc} & \otimes^2\mathbb{Z}[E] & \\ {\scriptstyle P=\bar{\sigma}}\swarrow & & \nwarrow{\scriptstyle H} \\ \hat{\otimes}^2\mathbb{Z}[E] & \xrightarrow[\partial]{\qquad} & \langle E\rangle_{\mathrm{nil}} \end{array} \right),$$

where $\bar{\sigma}$ is the natural projection in (2.7) and $\partial(a\hat{\otimes}b) = -a-b+a+b$ for $a, b \in E$. The weak equivalence

$$\overline{\mathbb{Z}}_{\mathrm{nil}}[E] \longrightarrow \Pi_{n,*}(\vee_E S^n), \tag{2.9}$$

sends $e \in E \subset \langle E\rangle_{\mathrm{nil}} = (\overline{\mathbb{Z}}_{\mathrm{nil}}[E])_0$ to the inclusion of the factor $i_e \colon S^n \subset \vee_E S^n$ corresponding to the index e regarded as an element $i_e \in \Omega^n(\vee_E S^n) \subset \langle \Omega^n(\vee_E S^n)\rangle_{\mathrm{nil}} = \Pi_{n,0}(\vee_E S^n)$.

Remark 2.10. Considering maps $f \colon S^n \to X$ together with tracks of such maps to the trivial map, we introduced in [BM08] the secondary homotopy group $\pi_{n,*}X$, which is a groupoid for $n = 0$, a crossed module for $n = 1$, a reduced quadratic module for $n = 2$, and a stable quadratic module for $n \geq 3$. Let **squad** be the category of stable quadratic modules.

Then using the adjoint functors Ad_n of the forgetful functors ϕ_n as discussed in [BM08, 6] we get the *additive secondary homotopy group* track functor

$$\Pi_{n,*} \colon \mathbf{Top}^* \longrightarrow \mathbf{squad}$$

given by

$$\Pi_{n,*}X = \begin{cases} \pi_{n,*}X, & \text{for } n \geq 3, \\ \mathsf{Ad}_3\pi_{2,*}X, & \text{for } n = 2, \\ \mathsf{Ad}_3\mathsf{Ad}_2\pi_{1,*}X, & \text{for } n = 1, \\ \mathsf{Ad}_3\mathsf{Ad}_2\mathsf{Ad}_1\pi_{0,*}X, & \text{for } n = 0. \end{cases}$$

This is the secondary analogue of (2.1).

Here the category **squad** of stable quadratic modules is not appropriate to study the smash product of secondary homotopy groups since we do not have a symmetric monoidal structure in **squad**. Therefore we introduced above the category **qpm** of quadratic pair modules and we observe that $\Pi_{n,*}X$ in **squad** yields a functor to the category **qpm** as follows. A map $f \colon X \to Y$ in $\mathbf{Top}^*$ induces a homomorphism $\Pi_{n,0}f \colon \Pi_{n,0}X \to \Pi_{n,0}Y$ between free nil-groups which carries generators in $\Pi_{n,0}X$ to generators in $\Pi_{n,0}Y$ and therefore $\Pi_{n,*}f$ is compatible with H. This shows that

Lemma 1.17 gives rise to a canonical lift

$$\begin{array}{ccc} & & \mathbf{qpm} \\ & \overset{\Pi_{n,*}}{\nearrow} & \downarrow \\ \mathbf{Top}^* & \xrightarrow[\Pi_{n,*}]{} & \mathbf{squad} \end{array}$$

Here the vertical arrow, which is the forgetful track functor given by (1.8), is faithful over the full subcategory spanned by 0-free quadratic pair modules, which includes additive secondary homotopy groups, but not full at the level of morphisms.

The definition of $\Pi_{2,*}X$ given above coincides with the lifting of $\mathsf{Ad}_3\pi_{2,*}X$ to **qpm** by the claim (*) in the proof of [BM08, 4.9]. Moreover, using [BM08, 4.16] and the definition of the functors Ad_n in [BM08, 6] one can easily check that the weak equivalence (2.9) is available in all dimensions $n \geq 0$.

In this paper we are concerned with the properties of the track functor $\Pi_{n,*}$, mapping to the category **qpm**. The category **qpm** is, in fact, a symmetric monoidal category, defined by a tensor product $\odot$ in **qpm**, see [BJP05], and the smash product yields the operator

$$\wedge\colon \Pi_{n,*}X \odot \Pi_{m,*}Y \longrightarrow \Pi_{n+m,*}(X \wedge Y) \tag{2.11}$$

constructed in [BM07]. Equation (2.3) has now a secondary analogue given by the right action of the symmetric group $\mathrm{Sym}(n+m)$ on the object $\Pi_{n+m,*}(X \wedge Y)$ in **qpm**. More precisely the following diagram commutes in **qpm**.

$$\begin{array}{ccc} \Pi_{n,*}X \odot \Pi_{m,*}Y & \xrightarrow{\wedge} & \Pi_{n+m,*}(X \wedge Y) \\ & & \cong \big\downarrow (\tau_{X,Y})_* \\ \tau_\odot \big\downarrow \cong & & \Pi_{n+m,*}(Y \wedge X) \\ & & \cong \big\uparrow \tau^*_{n,m} \\ \Pi_{m,*}Y \odot \Pi_{n,*}X & \xrightarrow{\wedge} & \Pi_{m+n,*}(Y \wedge X) \end{array}$$

Here $\tau_\odot$ on the left hand side is given by the symmetry of the tensor product $\odot$ in **qpm** and $\tau^*_{n,m}$ is defined by the action of $\mathrm{Sym}(n+m)$. For this reason we define and study in this paper the properties of the symmetric group action on secondary homotopy groups.

3 Actions of monoid-groupoids in track categories

In this paper we deal with actions on additive secondary homotopy groups. Additive secondary homotopy groups are objects in a track category. In ordinary categories a monoid action is given by a monoid-morphism mapping to an endomorphism monoid in the category. In track categories endomorphism objects are monoids in the monoidal category of groupoids, where the monoidal structure is given by the (cartesian) product. Therefore one can define accordingly actions of such monoids. We make explicit this structure in the following definition.

Definition 3.1. Let $\mathbb{I}$ be the category with only one object $*$ and one morphism $1\colon * \to *$. A *monoid-groupoid* $\mathbf{G}$ is a groupoid together with a multiplication functor $\cdot\colon \mathbf{G}\times\mathbf{G}\to\mathbf{G}$ and a unit functor $u\colon \mathbb{I}\to\mathbf{G}$, satisfying the laws of a monoid in the symmetric monoidal category of groupoids. We usually identify $* = u(*)$. The opposite $\mathbf{G}^{\mathrm{op}}$ of a monoid-groupoid is the underlying groupoid $\mathbf{G}$ with its unit functor and multiplication functor given by

$$\mathbf{G}\times\mathbf{G}\xrightarrow{T}\mathbf{G}\times\mathbf{G}\xrightarrow{\cdot}\mathbf{G}.$$

Here T is the interchange of factors in the product. A monoid-groupoid *morphism* $f\colon \mathbf{G}\to\mathbf{H}$ is a functor preserving the multiplication and the unit.

Monoid-groupoids are also termed strict monoidal groupoids. The weaker versions of this concept will not be considered in this paper, therefore we abbreviate the terminology.

The canonical example of a monoid-groupoid is obtained by the endomorphisms of an object X in a track category $\mathbf{C}$, denoted by

$$\mathbf{End}_{\mathbf{C}}(X).$$

The multiplication is given by composition in $\mathbf{C}$, and the unit is given by the identity morphism $1_X\colon X\to X$. In fact a monoid-groupoid as defined above is exactly the same thing as a track category with only one object, the opposite monoid-groupoid coincides with the the opposite of the corresponding track category and monoid-groupoid morphisms correspond to 2-functors.

The main example of endomorphism monoid-groupoid in this paper will be the one obtained from $\mathbf{C}=\mathbf{Top}^*$ the track category of pointed spaces and $X=S^n$ the n-dimensional sphere. This coincides with the fundamental groupoid of the topological monoid $\mathrm{End}_*(S^n)$ of pointed maps $S^n\to S^n$, see the next section.

Definition 3.2. Let X be an object in a track category $\mathbf{C}$ and let $\mathbf{G}$ be a monoid-groupoid. A *right action* of $\mathbf{G}$ on X is a monoid-groupoid morphism $\mathbf{G}^{\mathrm{op}}\to\mathbf{End}_{\mathbf{C}}(X)$.

The main goal of this paper is to construct right actions of the endomorphism monoid-groupoid of S^n on the additive secondary homotopy groups. This is achieved in Theorem 4.1.

Another important example of monoid-groupoid arises from crossed modules. The monoid-groupoid $M(\partial)$ associated to a crossed module $\partial\colon T\to G$ has object set G and morphism set the semidirect product $G\ltimes T$. Here we write the groups T and G with a multiplicative group law. An element $(g,t)\in G\ltimes T$ is a morphism $(g,t)\colon g\cdot\partial(t)\to g$ in $M(\partial)$. The composition law $\circ$ is given by the formula $(g,t)\circ(g\cdot\partial(t),t')=(g,t\cdot t')$. Multiplication in the groups G and $G\ltimes T$ defines the multiplication of $M(\partial)$ and the unit is given by the unit elements in G and $G\ltimes T$. Indeed this correspondence determines an equivalence between crossed modules and group objects in the category of groupoids. This example can be used to define crossed module actions.

Definition 3.3. Let X be an object in a track category $\mathbf{C}$ and let $\partial\colon T\to G$ be a crossed module. A *right action* of $\partial\colon T\to G$ on X is a monoid-groupoid morphism $M(\partial)^{\mathrm{op}}\to\mathbf{End}_{\mathbf{C}}(X)$.

We are interested in right actions of crossed modules in the track category **wqpm**. We explicitly describe such actions as follows.

Definition 3.4. A *right action of a crossed module* $\partial\colon T \to G$ on a quadratic pair module C in the category **wqpm** consists of a group action of G on the right of C given by morphisms in **wqpm**,

$$g^*\colon C \longrightarrow C, \;\; g \in G,$$

together with a bracket

$$\langle\!\langle -, - \rangle\!\rangle\colon C_0 \times T \longrightarrow C_1$$

satisfying the following properties, $x, y \in C_0$, $z \in C_1$, $s, t \in T$, $g \in G$,

1. $\langle\!\langle x + y, t \rangle\!\rangle = \langle\!\langle x, t \rangle\!\rangle^{\partial(t)^* y} + \langle\!\langle y, t \rangle\!\rangle$,
2. $x = \partial(t)^* x + \partial\langle\!\langle x, t \rangle\!\rangle$,
3. $z = \partial(t)^* z + \langle\!\langle \partial(z), t \rangle\!\rangle$,
4. $\langle\!\langle x, s \cdot t \rangle\!\rangle = \langle\!\langle \partial(s)^* x, t \rangle\!\rangle + \langle\!\langle x, s \rangle\!\rangle = \partial(t)^* \langle\!\langle x, s \rangle\!\rangle + \langle\!\langle x, t \rangle\!\rangle$,
5. $\langle\!\langle x, t^g \rangle\!\rangle = g^* \langle\!\langle (g^{-1})^* x, t \rangle\!\rangle$.

We point out that the second equality in (4) follows from (1)–(3). Indeed these are the two possible definitions of the horizontal composition $\langle\!\langle -, t \rangle\!\rangle \langle\!\langle -, s \rangle\!\rangle\colon \partial(st)^* \Rightarrow 1$ of the tracks $\langle\!\langle -, t \rangle\!\rangle\colon \partial(t)^* \Rightarrow 1$ and $\langle\!\langle -, s \rangle\!\rangle\colon \partial(s)^* \Rightarrow 1$ in the track category **wqpm**.

The notion of action defined above corresponds to an action in Norrie's sense ([Nor90]) of a crossed module on the underlying crossed module of a quadratic pair module, however Norrie considers left actions.

In this paper we will be interested in the action of some crossed modules arising from an algebraic structure that we call sign group, see Definition 3.5. This will be used to describe the symmetric action on additive secondary homotopy groups in Section 5.

Definition 3.5. Let $\{\pm 1\}$ be the multiplicative group of order 2. A *sign group* $G_\Box$ is a diagram of group homomorphisms

$$\{\pm 1\} \overset{\imath}{\hookrightarrow} G_\Box \overset{\delta}{\twoheadrightarrow} G \overset{\varepsilon}{\longrightarrow} \{\pm 1\}$$

where the first two morphisms form a central extension. Here all groups have a multiplicative group law and the composite $\varepsilon\delta$ is also denoted by $\varepsilon\colon G_\Box \to \{\pm 1\}$. Moreover, we define the element $\omega = \imath(-1) \in G_\Box$.

A sign group $G_\Box$ *acts on the right of a quadratic pair module* C if G acts on the right of C by morphisms

$$g^*\colon C \longrightarrow C, \;\; g \in G, \quad \text{in } \mathbf{qpm},$$

and there is a bracket

$$\langle -, - \rangle\colon C_0 \times G_\Box \longrightarrow C_1$$

satisfying the following properties, $x, y \in C_0$, $z \in C_1$, $s, t \in G_\Box$, were $\varepsilon(t)^*$ is given by the action of $(\mathbb{Z}, \cdot)$ in Definition 1.6,

1. $\langle x+y,t\rangle = \langle x,t\rangle^{\delta(t)^*y} + \langle y,t\rangle$,
2. $\varepsilon(t)^*(x) = \delta(t)^*(x) + \partial\langle x,t\rangle$,
3. $\varepsilon(t)^*(z) = \delta(t)^*(z) + \langle \partial(z),t\rangle$,
4. $\langle x, s\cdot t\rangle = \langle \delta(s)^*(x),t\rangle + \langle \varepsilon(t)^*x, s\rangle$,
5. the *ω-formula*:
$$\langle x,\omega\rangle = P(x|x)_H.$$

Notice that the ω-formula corresponds to the k-invariant, see [BM08, 8].

Remark 3.6. A sign group $G_\square$ gives rise to a crossed module

$$\delta_\square = (\varepsilon,\delta)\colon G_\square \longrightarrow \{\pm 1\}\times G,$$

where $\{\pm1\}\times G$ acts on $G_\square$ by the formula

$$g^{(x,h)} = \bar{h}^{-1} g \bar{h}\imath\left(\varepsilon(g)^{\binom{x\cdot\varepsilon(h)}{2}}\right).$$

Here $g\in G_\square$, $x\in\{\pm1\}$, $h\in G$ and $\bar{h}\in G_\square$ is any element with $\delta(\bar{h}) = h$. This action is well defined since $G_\square$ is a central extension of G by $\{\pm1\}$.

Lemma 3.7. *The sign group action in Definition 3.5 corresponds to an action of the crossed module $\delta_\square$ on C in the sense of Definition 3.4 such that $\{\pm1\}$ acts on C by the action of $(\mathbb{Z},\cdot)$ in Definition 1.6, G acts by morphisms in* **qpm***, and the ω-formula holds. The correspondence is given by the formula*

$$\langle\langle x,t\rangle\rangle = \langle \varepsilon(t)^*x, t\rangle,\quad x\in C_0,\quad t\in G_\square.$$

The proof of this lemma is straightforward. We just want to point out that Definition 3.4 (5) follows in this case from Definition 3.5 (4), (5), and Lemma 1.18.

Remark 3.8. A sign group $G_\square$ is *trivial* if G is a trivial group. Notice that a trivial sign group acts on any quadratic pair module in a unique way.

The main examples of sign groups considered in this paper are the symmetric track groups in Section 5, which act on the additive secondary homotopy groups.

4 The action of $\mathrm{End}(S^n)$ on $\Pi_{n,*}X$

Let S^n be the n-sphere and let $\mathrm{End}_*(S^n) = \Omega^n S^n$ be the topological monoid of maps $S^n\to S^n$ in $\mathbf{Top}^*$. Then the fundamental groupoid of $\mathrm{End}_*(S^n)$, denoted by $\pi_{0,*}\mathrm{End}_*(S^n)$, is a monoid-groupoid in the sense of Definition 3.1. This monoid-groupoid coincides with the endomorphism monoid-groupoid of S^n in the track category $\mathbf{Top}^*$. It is well known that the monoid of path components of $\mathrm{End}_*(S^n)$ coincides with the multiplicative monoid $(\mathbb{Z},\cdot)$.

We now consider the right action of $\pi_{0,*}\mathrm{End}_*(S^n)$ on $\Pi_{n,*}X$ for $n\geq 2$. That is, we define for each pointed map $f\colon S^n\to S^n$ an induced map in **qpm**

$$f^*\colon \Pi_{n,*}X \longrightarrow \Pi_{n,*}Y$$

and we define for each track $H\colon f \Rightarrow g$ with $f, g\colon S^n \to S^n$ a track in **qpm**

$$H^*\colon f^* \Rightarrow g^*.$$

This yields a right action of the monoid-groupoid $\pi_{0,*}\operatorname{End}_*(S^n)$ on the secondary homotopy group $\Pi_{n,*}X$ in the track category **qpm** of quadratic pair modules in the sense of Definition 3.2.

Theorem 4.1. *Let X be a pointed space. For any $n \geq 2$ there is a natural action of the monoid-groupoid $\pi_{0,*}\operatorname{End}_*(S^n)$ on the quadratic pair module $\Pi_{n,*}X$.*

The rest of this section is devoted to the proof of this theorem, which is carried out in several steps.

The discrete monoid $\pi_{0,0}\operatorname{End}_*(S^n)$, which is the underlying set of the topological monoid $\operatorname{End}_*(S^n)$, acts on the right of the pointed discrete set $\Omega^n X$ of pointed maps $S^n \to X$ by precomposition, i. e. given $f\colon S^n \to S^n$ the induced endomorphism is

$$f^*\colon \Omega^n X \longrightarrow \Omega^n X, \quad f^*(g) = gf.$$

This induces a right action of $\pi_{0,0}\operatorname{End}_*(S^n)$ on the free group $\pi_{n,*}X = \langle \Omega^n X \rangle_{\mathrm{nil}}$ of nilpotency class 2 which will be denoted in the same way.

In order to extend this action to $\Pi_{n,1}X$ we consider the submonoid

$$\tilde{\pi}_{0,1}\operatorname{End}_*(S^n) \subset \pi_{0,1}\operatorname{End}_*(S^n) \tag{4.2}$$

of the monoid $\pi_{0,1}\operatorname{End}_*(S^n)$ of morphisms in $\pi_{0,*}\operatorname{End}_*(S^n)$ given by tracks between self-maps of S^n of the form

$$\gamma\colon f \Rightarrow \Sigma^{n-1}(\cdot)^{\deg f} = (\cdot)_n^{\deg f}. \tag{4.3}$$

Here $\deg f \in \mathbb{Z}$ denotes the degree of $f\colon S^n \to S^n$ and for $k \in \mathbb{Z}$

$$(\cdot)^k\colon S^1 \longrightarrow S^1\colon z \mapsto z^k$$

is given by the (multiplicative) topological abelian group structure of S^1.

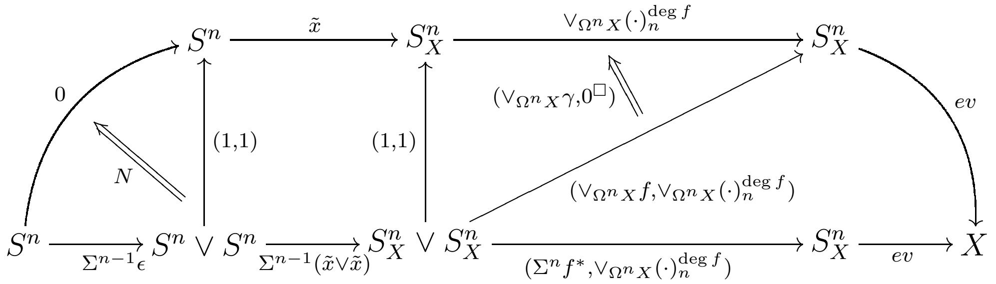
We need a bracket operation

$$\langle -, - \rangle\colon \Pi_{n,0}X \times \tilde{\pi}_{0,1}\operatorname{End}_*(S^n) \longrightarrow \Pi_{n,1}X, \tag{4.4}$$

defined as follows. Let $x \in \pi_{n,0}X = \langle \Omega^n X \rangle_{\mathrm{nil}}$ and $\gamma\colon f \Rightarrow (\cdot)_n^{\deg f}$ in $\tilde{\pi}_{0,1}\operatorname{End}_*(S^n)$. We choose maps $\tilde{x}\colon S^1 \to \vee_{\Omega^n X} S^1$, $\epsilon\colon S^1 \to S^1 \vee S^1$ with $(\pi_1\tilde{x})_{\mathrm{nil}}(1) = x$ and $(\pi_1\epsilon)_{\mathrm{nil}} = -a + b \in \pi_1(S^1 \vee S^1)_{\mathrm{nil}}$, the difference between the inclusion of the first and the second factor of $S^1 \vee S^1$. Then $\langle x, \gamma \rangle \in \Pi_{n,1}X$ is the element represented by the map

$$S^1 \xrightarrow{\epsilon} S^1 \vee S^1 \xrightarrow{\tilde{x}\vee\tilde{x}} (\vee_{\Omega^n X} S^1) \vee (\vee_{\Omega^n X} S^1) \xrightarrow{(\Sigma f^*, \vee_{\Omega^n X}(\cdot)^{\deg f})} \vee_{\Omega^n X} S^1$$

and the track



(4.5)

Here N is a nil-track.

The main properties of the bracket operation in (4.4) are listed in the following proposition.

Proposition 4.6. *The bracket $\langle -,-\rangle$ in (4.4) satisfies the following formulas for any $x,y\in \Pi_{n,0}X$ and $\gamma\colon f\Rightarrow (\cdot)_n^{\deg f}, \delta\colon g\Rightarrow (\cdot)_n^{\deg g}$ in $\tilde{\pi}_{0,1}\operatorname{End}_*(S^n)$.*

1. $\langle x+y,\gamma\rangle = \langle x,\gamma\rangle^{f^*y}+\langle y,\gamma\rangle$,
2. $(\deg f)^*x = f^*x+\partial\langle x,\gamma\rangle$,
3. $\langle x,\gamma\delta\rangle = \langle f^*x,\delta\rangle + \langle(\deg g)^*x,\gamma\rangle$,
4. *if $\omega\colon 1_{S^n}\Rightarrow 1_{S^n}$ is a track with $0\neq \overline{Hopf}(\omega)\in\hat{\otimes}^2\mathbb{Z}=\mathbb{Z}/2$ then $\langle x,\omega\rangle = P(x|x)_H$.*

Moreover, this bracket operation is natural in X.

Proof. With the notation in [BM08, 7.4] we have $\langle x,\gamma\rangle = r(ev(\vee_{\Omega^nX}\gamma)(\Sigma^{n-1}\tilde{x}))$ for the track

$$ev(\vee_{\Omega^nX}\gamma)(\Sigma^{n-1}\tilde{x})\colon ev(\Sigma^nf^*)(\Sigma^{n-1}\tilde{x}) = ev(\vee_{\Omega^nX}f)(\Sigma^{n-1}\tilde{x})\Rightarrow ev(\vee_{\Omega^nX}(\cdot)_n^{\deg f})(\Sigma^{n-1}\tilde{x}),$$

therefore (1) and (2) follow from [BM08, 7.6 and 7.5 (2)].

It is easy to see that the formula

$$ev(\vee_{\Omega^nX}\gamma\delta) = (ev(\vee_{\Omega^nX}\gamma)(\vee_{\Omega^nX}(\cdot)_n^{\deg g}))\square(ev(\vee_{\Omega^nX}\delta)(\Sigma^nf^*))$$

holds, therefore (3) follows from [BM08, 7.5 (3)].

If we evaluate $\langle x,-\rangle$ at ω then the composite track obtained from (4.5) by going from the lower left S^n to the upper right S^n_X has the same reduced Hopf invariant as the track from S^n to S^n_X in (2.8). Indeed the formula for both reduced Hopf invariants is (c) in the proof of Proposition 4.8. Therefore (4) follows. ■

The next result follows from the algebraic properties of the bracket (4.4) which are proved in the previous proposition together with Lemma 1.18.

Proposition 4.7. *The monoid $\tilde{\pi}_{0,1}\operatorname{End}_*(S^n)$ acts on the right of $\Pi_{n,1}X$ by the following formula, $n\geq 2$: given $x\in\Pi_{n,1}X$ and $\gamma\colon f\Rightarrow(\cdot)_n^{\deg f}$*

$$\gamma^*x = (\deg f)^*x - \langle\partial(x),\gamma\rangle.$$

This action satisfies $\partial\gamma^* = f^*\partial$, $\gamma^*P = P(\otimes^2 f^*_{\mathrm{ab}})$, and $Hf^* = (\otimes^2 f^*_{\mathrm{ab}})H$, therefore it defines an action of $\tilde{\pi}_{0,1}\operatorname{End}_*(S^n)$ on the right of the quadratic pair module $\Pi_{n,*}X$ in the category **qpm**. *This action is natural in X.*

Proof. The equality $Hf^* = (\otimes^2 f^*_{\mathrm{ab}})H$ follows from the fact that the endomorphism f^* carries generators to generators in $\langle\Omega^nX\rangle_{\mathrm{nil}}$. The equality $\partial\gamma^* = f^*\partial$ follows from Proposition 4.6 (2). Let us check $\gamma^* P = P(\otimes^2 f^*_{\mathrm{ab}})$. Given $a, b \in \Omega^n X$

$$\begin{aligned}
\gamma^* P(a \otimes b) &= (\deg f)^* P(a \otimes b) - \langle -a - b + a + b, \gamma \rangle \\
&= P(\deg f)^2 (a \otimes b) - \langle b, \gamma \rangle - \langle a, \gamma \rangle + \langle b, \gamma \rangle + \langle a, \gamma \rangle \\
&\quad + P(-f^*(a) + (\deg f)^* a | f^* b)_H - P(-f^*(b) + (\deg f)^* b | f^* a)_H \\
&= P(\deg f)^2 (a \otimes b) + P(\partial \langle b, \gamma \rangle | \partial \langle a, \gamma \rangle)_H \\
&\quad + P(-f^*(a) + (\deg f)^* a | f^* b)_H - P(-f^*(b) + (\deg f)^* b | f^* a)_H \\
&= -P(\deg f)^2 (a | b)_H \\
&\quad - P(-f^*(a) + (\deg f)^* a | - f^*(b) + (\deg f)^* b)_H \\
&\quad + P(-f^*(a) + (\deg f)^* a | f^* b)_H + P(f^* a | - f^*(b) + (\deg f)^* b)_H \\
&= -P(f^*(a) | f^*(b))_H \\
&= P(f^*(b) | f^*(a))_H \\
&= P(f^*(a) \otimes f^*(b)).
\end{aligned}$$

Here we use Proposition 4.6 (1) and (2) and the fact that $H(a) = 0 = H(b)$.

Finally given $\delta \colon g \Rightarrow (\cdot)_n^{\deg g}$

$$\begin{aligned}
\gamma^* \delta^*(x) &= \gamma^*((\deg g)^* x - \langle \partial(x), \delta \rangle) \\
&= (\deg f)^* (\deg g)^* x - (\deg f)^* \langle \partial(x), \delta \rangle - \langle g^* \partial(x), \gamma \rangle \\
&= ((\deg f)(\deg g))^* x - \langle (\deg f)^* \partial(x), \delta \rangle - \langle g^* \partial(x), \gamma \rangle \\
&\quad + \binom{\deg f}{2} \binom{\deg g}{2} P(\partial(x) | \partial(x))_H \\
&= (\deg fg)^* x - \langle \partial(x), \gamma \delta \rangle \\
&= (\gamma \delta)^*(x).
\end{aligned}$$

Here we use Proposition 4.6 (1), (2) and (3), Lemma 1.18 and the fact that $P(\partial(x)|\partial(x))_H = -x - x + x + x = 0$. ∎

Proposition 4.8. *For $n \geq 2$ the right action of the monoid $\tilde{\pi}_{0,1} \operatorname{End}_*(S^n)$ on the group $\Pi_{n,1} X$ given by Proposition 4.7 factors through the boundary homomorphism*

$$q \colon \tilde{\pi}_{0,1} \operatorname{End}_*(S^n) \twoheadrightarrow \pi_{0,0} \operatorname{End}_*(S^n), \quad q(\gamma \colon f \Rightarrow (\cdot)_n^{\deg f}) = f,$$

that is, the homomorphism $\gamma^* = f^*$ only depends on the boundary $q(\gamma) = f$.

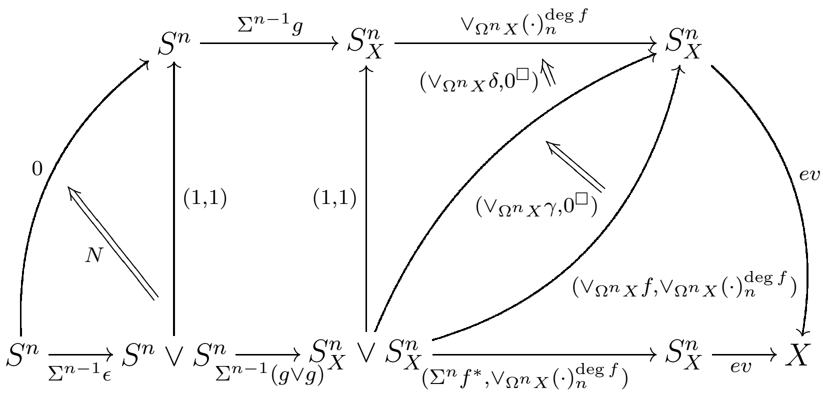
Proof. Let $\gamma \colon f \Rightarrow (\cdot)_n^{\deg f}$ be any element in $\tilde{\pi}_{0,1} \operatorname{End}_*(S^n)$ and let $\delta \colon (\cdot)_n^{\deg f} \Rightarrow (\cdot)_n^{\deg f}$ be any track. We know that all elements in $q^{-1}(f)$ are of the form $\delta \square \gamma$, therefore we only have to check that for any $[g, G] \in \Pi_{n,1} X$

$$\gamma^*[g, G] = (\delta \square \gamma)^*[g, G],$$

or equivalently

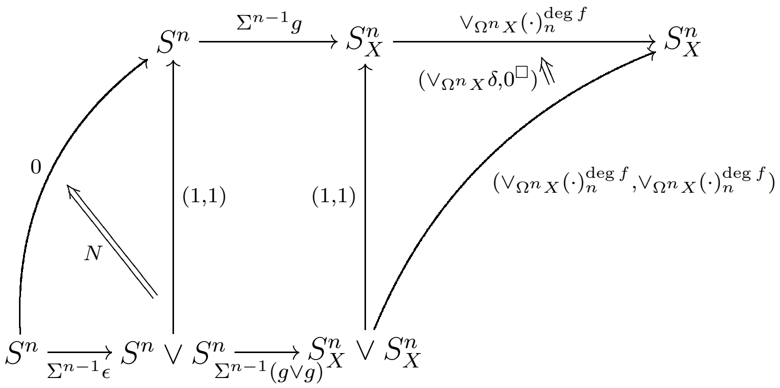
$$\langle \partial[g, G], \gamma \rangle = \langle \partial[g, G], \delta \square \gamma \rangle.$$

By definition of the bracket $\langle -, - \rangle$ in (4.4), see diagram (4.5), the element $\langle \partial[g, G], \delta\Box\gamma \rangle$ is represented by the following diagram



(a)

Let us now pay special attention to the following subdiagram of (a)



(b)

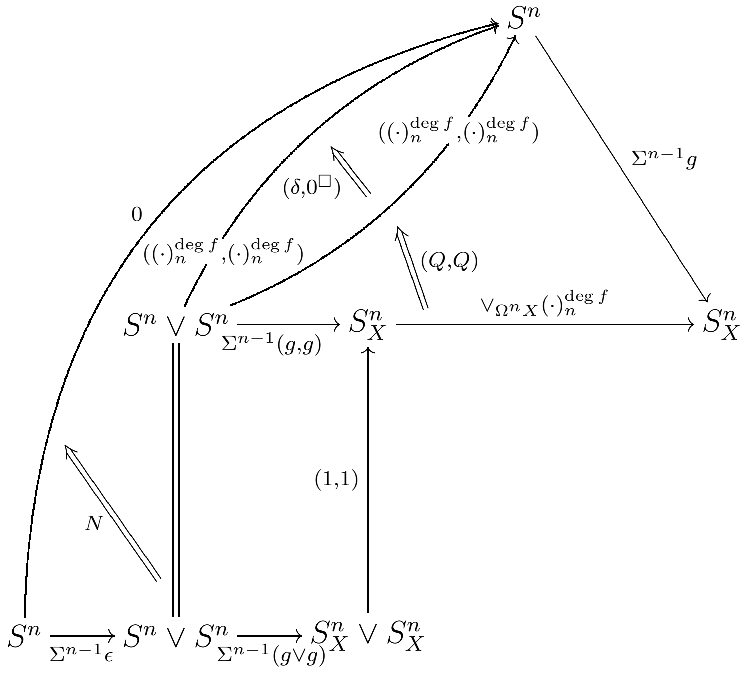
This is a composite track, termed (b), between $(n-1)$-fold suspensions. By the elementary properties of the Hopf invariant for tracks [BM08, 3] the reduced Hopf invariant of (b) is trivial provided $\overline{Hopf}(\delta) = 0$. Again by [BM08, 3] if $0 \neq \overline{Hopf}(\delta) \in \hat{\otimes}^2\mathbb{Z} = \mathbb{Z}/2$ then the reduced Hopf invariant of (b) is given by the formula below. For the formula we need to fix a notation for the linear expansion of $(\pi_1 g)_{\mathrm{ab}}(1) \in \pi_1(\vee_{\Omega^n X} S^1)_{\mathrm{ab}} = \mathbb{Z}[\Omega^n X]$ in terms of generators $a_i \in \Omega^n X$ and $n_i \in \mathbb{Z}$, $(\pi_1 g)_{\mathrm{ab}}(1) = \sum_{i=0}^k n_i a_i$.

$$\overline{Hopf}(\mathrm{b}) = \sum_{i=0}^{k} n_i a_i \hat{\otimes} a_i \in \hat{\otimes}^2 \mathbb{Z}[\Omega^n X]. \qquad \text{(c)}$$

By using once again the elementary properties of the Hopf invariant for tracks described in [BM08, 3] the reader can easily check that for any track

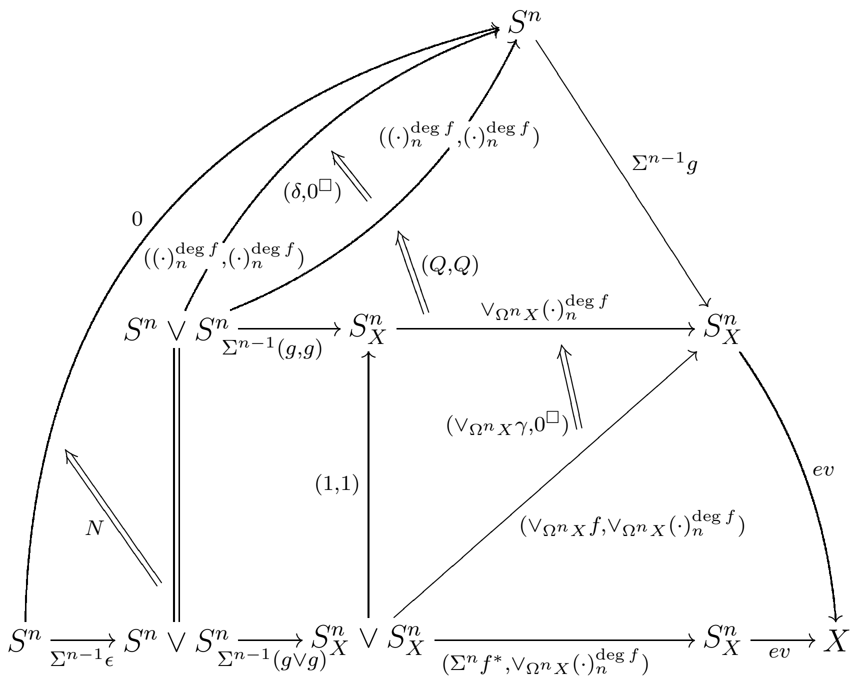
$$Q\colon (\vee_{\Omega^n X}(\cdot)^{\deg f}_n)(\Sigma^{n-1}g) \Rightarrow (\Sigma^{n-1}g)(\cdot)^{\deg f}_n$$

the following composite track has the same reduced Hopf invariant as (b)



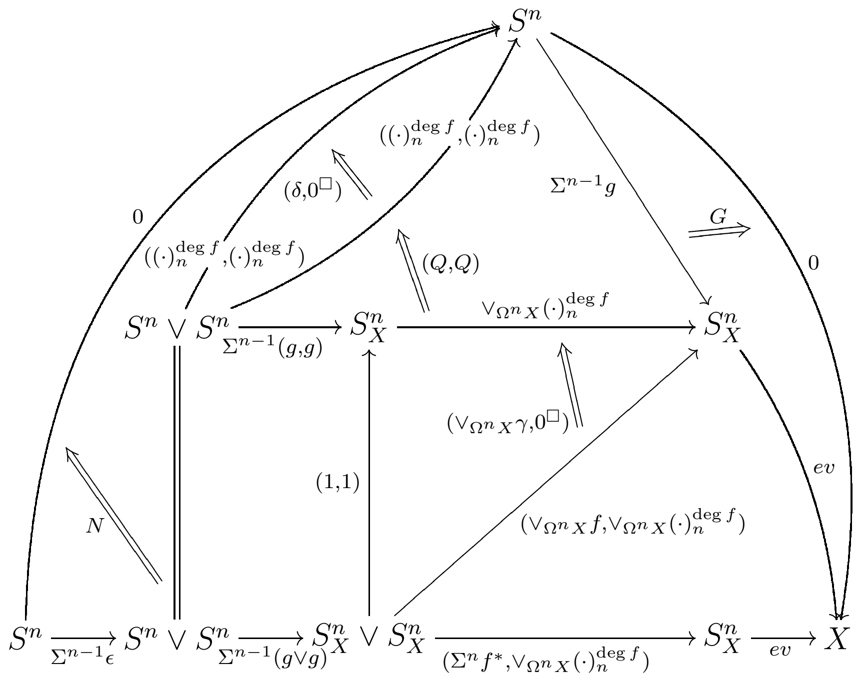
(d)

Since (b) and (d) have the same reduced Hopf invariant then we can replace subdiagram (b) in (a) by (d), obtaining the same element in $\Pi_{n,1}X$, namely



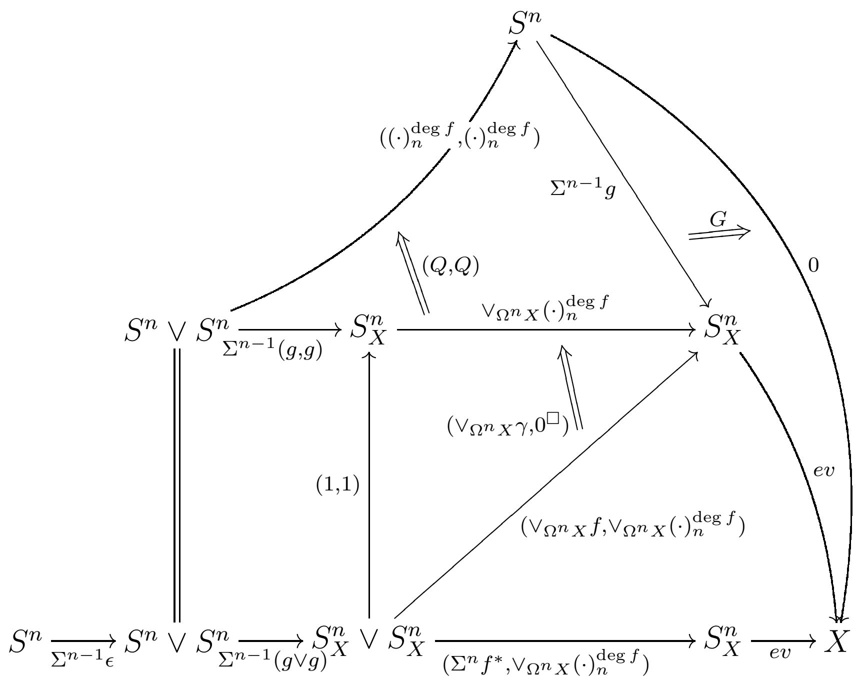
(e)

Pasting the trivial track $G0$ (see Remark 1.9) to (e) gives



(f)

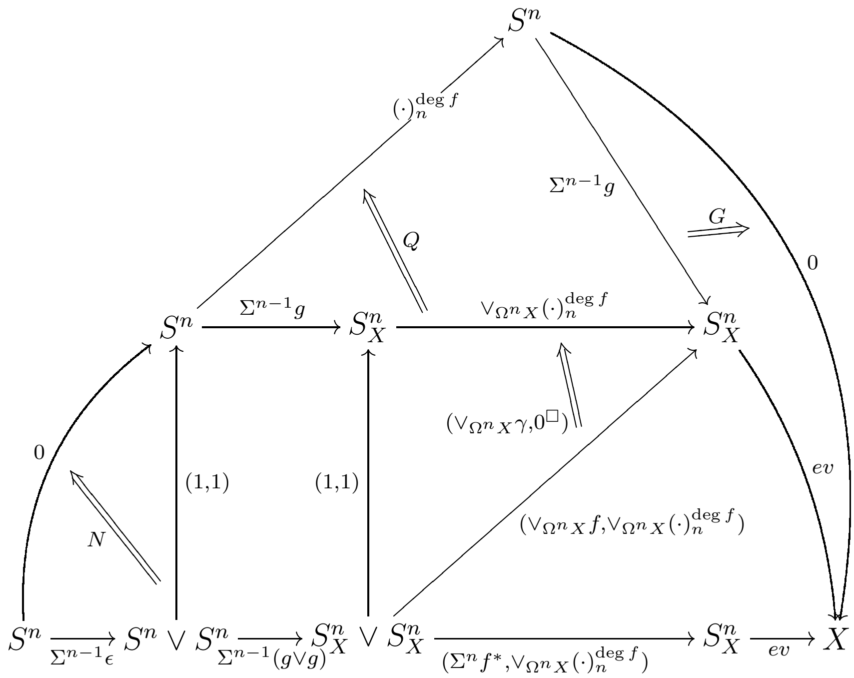
Again by Remark 1.9 we can remove some trivial tracks from (f).



(g)

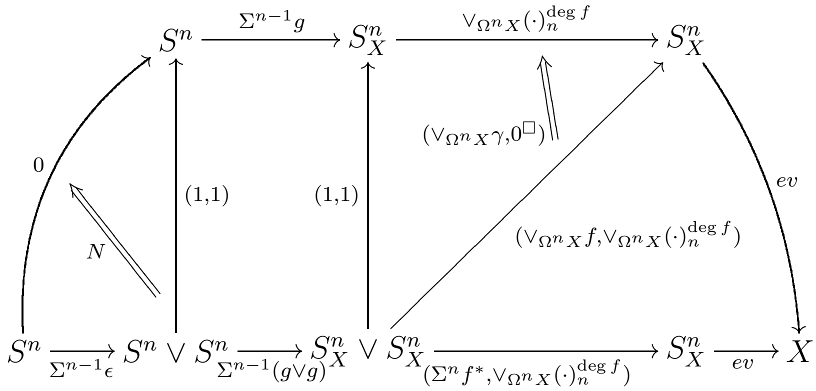
Pasting another trivial track to (g) and factoring some maps and tracks through

$(1,1)\colon S^n \vee S^n \to S^n$ we obtain



(h)

Finally removing trivial tracks from (h) gives



(i)

Notice that this last composite track (i) represents $\langle \partial[g,G], \gamma\rangle$, see (4.5), hence we are done. ∎

The next corollary follows from the two previous propositions.

Corollary 4.9. *For any pointed space X and $n \geq 2$ the monoid $\pi_{0,0}\operatorname{End}_*(S^n)$ acts on the right of the quadratic pair module $\Pi_{n,*}X$. This action is natural in X.*

Now Theorem 4.1 is a consequence of the next result.

Proposition 4.10. *The action of $\pi_{0,0}\operatorname{End}_*(S^n)$ on the right of $\Pi_{n,*}X$ given by Corollary 4.9 extends to an action of the whole monoid-groupoid $\pi_{0,*}\operatorname{End}_*(S^n)$, $n \geq 2$.*

Proof. A morphism H in $\pi_{0,*}\operatorname{End}_*(S^n)$ is a track $H\colon f \Rightarrow g$ between maps $f,g\colon S^n \to S^n$, in particular $\deg f = \deg g = k \in \mathbb{Z}$. In order to define a track

$$H^*\colon f^* \Rightarrow g^*$$

between the quadratic pair module morphisms

$$f^*, g^* \colon \Pi_{n,*}X \longrightarrow \Pi_{n,*}X$$

we choose tracks in $\tilde{\pi}_{0,1}\,\mathrm{End}_*(S^n)$

$$\alpha\colon f \Rightarrow (\cdot)^k_n,$$
$$\beta\colon g \Rightarrow (\cdot)^k_n,$$

such that

$$H = \beta^{\boxminus}\Box\alpha.$$

By Proposition 4.6 the maps

$$\langle -,\alpha\rangle, \langle -,\beta\rangle \colon \Pi_{n,0}X \longrightarrow \Pi_{n,1}X$$

are tracks

$$\langle -,\alpha\rangle\colon f^* \Rightarrow k^*,$$
$$\langle -,\beta\rangle\colon g^* \Rightarrow k^*,$$

in the category **wqpm**, therefore we can define H^* as the vertical composition

$$H^* = \langle -,\beta\rangle^{\boxminus}\Box\langle -,\alpha\rangle,$$

i. e. H^* is the map

$$H^*\colon \Pi_{n,0}X \longrightarrow \Pi_{n,1}X$$

defined by

$$H^*(x) = \langle x,\alpha\rangle - \langle x,\beta\rangle.$$

By the proof of Proposition 4.6 and by [BM08, 7.5 (3)] the element $H(x)$ coincides with $r(ev(\vee_{\Omega^n X}H)(\Sigma^{n-1}\tilde{x}))$ for $\tilde{x}\colon S^1 \to \vee_{\Omega^n X}S^1$ any map with $(\pi_1\tilde{x})_{\mathrm{nil}}(1) = x$ in the sense of [BM08, 7.4]. The reader can now use the properties of the bracket (4.4) described in Proposition 4.6 together with [BM08, 7.5 (3)] to check that this yields a monoid-groupoid action. ■

Later we will consider the quotient monoid $\bar{\pi}_{0,1}\,\mathrm{End}_*(S^2)$ of $\tilde{\pi}_{0,1}\,\mathrm{End}_*(S^2)$ defined as follows: two elements $\gamma\colon f \Rightarrow (\cdot)_2^{\deg f}$, $\bar{\gamma}\colon g \Rightarrow (\cdot)_2^{\deg g}$ in $\tilde{\pi}_{0,1}\,\mathrm{End}_*(S^2)$ represent the same element in $\bar{\pi}_{0,1}\,\mathrm{End}_*(S^2)$ provided $\deg f = \deg g$ and

$$0 = \overline{\mathit{Hopf}}(\bar{\gamma}\Box\gamma^{\boxminus}) \in \hat{\otimes}^2\mathbb{Z} = \mathbb{Z}/2.$$

Proposition 4.11. *The bracket operation (4.4) factors for $n = 2$ through the natural projection $\tilde{\pi}_{0,1}\,\mathrm{End}_*(S^2) \twoheadrightarrow \bar{\pi}_{0,1}\,\mathrm{End}_*(S^2)$.*

$$\langle -,-\rangle\colon \Pi_{2,0}X \times \bar{\pi}_{0,1}\,\mathrm{End}_*(S^2) \longrightarrow \Pi_{2,1}X.$$

Proof. Two tracks γ and $\bar{\gamma}$ in $\tilde{\pi}_{0,1}\,\mathrm{End}_*(S^2)$ represent the same element in $\bar{\pi}_{0,1}\,\mathrm{End}_*(S^2)$ if and only if $\bar{\gamma} = \delta\Box\gamma$ for some $\delta\colon (\cdot)_2^k \Rightarrow (\cdot)_2^k$ with $\overline{\mathit{Hopf}}(\delta) = 0$, so we only need to check that $\langle x,\gamma\rangle = \langle x,\delta\Box\gamma\rangle$. The element $\langle x,\delta\Box\gamma\rangle$ is represented by diagram (a) in the proof of Proposition 4.8 where we assume that g is a map with $(\pi_1 g)_{\mathrm{nil}}(1) = x$. As we mention in that proof diagram (b) is a nil-track in these circumstances, therefore we can drop δ from (a) and still obtain the same element in $\Pi_{2,1}X$. But if we drop δ we obtain $\langle x,\gamma\rangle$, hence we are done. ■

5 The symmetric action on secondary homotopy groups

The permutation of coordinates in $S^n = S^1 \wedge \cdots \wedge S^1$ induces a left action of the symmetric group $\mathrm{Sym}(n)$ on the n-sphere S^n. This action induces a monoid inclusion

$$\mathrm{Sym}(n) \subset \pi_{0,0}\,\mathrm{End}_*(S^n). \tag{5.1}$$

We define the *symmetric track group* for $n \geq 3$

$$\mathrm{Sym}_\square(n) \subset \tilde{\pi}_{0,1}\,\mathrm{End}_*(S^n)$$

as the submonoid of tracks of the from

$$\alpha\colon \sigma \Rightarrow (\cdot)_n^{\mathrm{sign}(\sigma)},$$

where $\sigma \in \mathrm{Sym}(n)$ and $\mathrm{sign}(\sigma) \in \{\pm 1\}$ is the sign of the permutation. Compare the notation in (1.7) and (4.3).

The submonoid defined as above for $n = 2$ will be called the *extended symmetric track group*

$$\overline{\mathrm{Sym}}_\square(2) \subset \tilde{\pi}_{0,1}\,\mathrm{End}_*(S^2).$$

For $n = 2$ the symmetric track group $\mathrm{Sym}_\square(2)$ is the image of $\overline{\mathrm{Sym}}_\square(2)$ by the natural projection $\tilde{\pi}_{0,1}\,\mathrm{End}_*(S^2) \twoheadrightarrow \bar{\pi}_{0,1}\,\mathrm{End}_*(S^2)$ in Proposition 4.11.

$$\mathrm{Sym}_\square(2) \subset \bar{\pi}_{0,1}\,\mathrm{End}_*(S^2). \tag{5.2}$$

Proposition 5.3. *The symmetric track group is indeed a group. Moreover, it fits into a central extension, $n \geq 2$,*

$$\mathbb{Z}/2 \hookrightarrow \mathrm{Sym}_\square(n) \overset{\delta}{\twoheadrightarrow} \mathrm{Sym}(n)$$

with $\delta(\alpha) = \sigma$, which splits if and only if $n = 2$ or 3.

This proposition follows from Corollary 6.9 and Remarks 6.10 and 6.12 below.

For $n = 0$ and $n = 1$ we define $\mathrm{Sym}(n)$ to be the trivial group, and $\mathrm{Sym}_\square(n)$ the trivial sign group. Then the symmetric track group $\mathrm{Sym}_\square(n)$ is a sign group ($n \geq 0$)

$$\{\pm 1\} \hookrightarrow \mathrm{Sym}_\square(n) \overset{\delta}{\twoheadrightarrow} \mathrm{Sym}(n) \xrightarrow{\mathrm{sign}} \{\pm 1\}$$

as in Definition 3.5.

Theorem 5.4. *Let X be a pointed space. For $n \geq 0$ the symmetric group $\mathrm{Sym}(n)$ acts naturally on the right of the additive secondary homotopy group $\Pi_{n,*}X$ in the category **qpm** of quadratic pair modules. Moreover, the restriction*

$$\langle -, - \rangle \colon \Pi_{n,0}X \times \mathrm{Sym}_\square(n) \longrightarrow \Pi_{n,1}X$$

of the bracket defined in (4.4) if $n \geq 3$ and in Proposition 4.11 if $n = 2$ yields a natural right action of the sign group $\mathrm{Sym}_\square(n)$ on $\Pi_{n,*}X$ in the sense of Definition 3.5.

The action of Sym(n) is given by Corollary 4.9 and the inclusion (5.1) if $n \geq 3$ or (5.2) if $n = 2$. The rest of the statement follows from Proposition 4.6. The cases $n = 0, 1$ are trivial consequences of Remark 3.8.

Remark 5.5. Computations with secondary homotopy groups can be carried out in a rather explicit way. We commented on an example from [BM08, 5.1] in (2.9). One can find further computations in [BM06]. In this remark we perform a computation which takes into account the action of symmetric track groups.

The aim of this remark is to provide a small model for the n^{th} additive secondary homotopy group of the sphere S^n equipped with an action of $\mathrm{Sym}_\square(n)$. The weak equivalence of quadratic pair modules (2.9) yields a small model of $\Pi_{n,*}S^n$ for all $n \geq 0$ given by

$$\overline{\mathbb{Z}}_{\text{nil}} = \left(\begin{array}{ccc} & \mathbb{Z} & \\ {}^{P}\swarrow & & \nwarrow^{H(n)=\binom{n}{2}} \\ \mathbb{Z}/2 & \xrightarrow[\partial=0]{} & \mathbb{Z} \end{array} \right).$$

However for $n \geq 2$ this algebraic model does not carry any action of $\mathrm{Sym}_\square(n)$, hence it is not a good model for $\Pi_{n,*}S^n$. We now compute a small model $A(\mathrm{Sym}_\square(n))$ with an action of $\mathrm{Sym}_\square(n)$. This model is obtained by a procedure which resembles the construction of the group-ring of a group G. As an abelian group this group-ring is $\mathbb{Z}[G_+]$, where G_+ denotes the group G with an outer base point. This is the universal way of extending the integers $\mathbb{Z}$ to an abelian group with an action of G.

For a sign group $G_\square$, as for example $G_\square = \mathrm{Sym}_\square(n)$, the quadratic pair module $A(G_\square)$ is going to be defined as the universal extension of $\overline{\mathbb{Z}}_{\text{nil}}$ to a weakly equivalent quadratic pair module with an action of $G_\square$. The extension will be given by a morphism of quadratic pair modules which we denote by

$$\xi\colon \overline{\mathbb{Z}}_{\text{nil}} \longrightarrow A(G_\square).$$

We now carefully construct $A(G_\square)$ and ξ step by step, by looking at the properties they must satisfy.

Recall from Definition 3.5 that a $G_\square$-action consists, first of all, of an action of the group G. The lower group of a quadratic pair module has always nilpotency class 2 and the universal extension of $(\overline{\mathbb{Z}}_{\text{nil}})_0 = \mathbb{Z}$ to a group of nilpotency class 2 with an action of G is

$$A_0(G_\square) \;=\; \langle G_+\rangle_{\text{nil}},$$

and the homomorphism ξ_0 sends $1 \in \mathbb{Z}$ to $1 \in G \subset \langle G_+\rangle_{\text{nil}}$.

The $G_\square$-action does not generate elements in $A_{ee}(G_\square)$, see Definition 3.5, and G acts on $A(G_\square)$ as a quadratic pair module, so given $g \in G \subset A_0(G_\square)$

$$H(g) \;=\; H(g^*1) \;=\; g^*H(1) \;=\; g^*\binom{1}{2} \;=\; g^*\frac{1(1-1)}{2} \;=\; g^*0 \;=\; 0,$$

so the action of G on $A_0(G_\square)$ takes generators to generators. This shows that $A(G_\square)$ must be 0-free, hence $A_{ee}(G_\square) = \otimes^2\mathbb{Z}[G_+]$ and H is as in Definition 1.16. In particular, $\xi_{ee} = \otimes^2(\xi_0)_{\text{ab}}$. Moreover, since $(\overline{\mathbb{Z}}_{\text{nil}})_1 = \mathbb{Z}/2$ is generated by $P(1|1)_H$

then ξ_1 will be fully determined by ξ_0 and the fact that the quadratic pair module morphism ξ must be compatible with P and H.

The computation of $A_1(G_\Box)$ is more complicated. There are essentially two sources of elements for this group, the central homomorphism

$$P(-|-)_H\colon \otimes^2 \mathbb{Z}[G_+] \longrightarrow A_1(G_\Box),$$

which is part of the stable quadratic module (1.8), and the bracket operation of the $G_\Box$-action,

$$\langle -,-\rangle\colon A_0(G_\Box)\times G_\Box = \langle G_+\rangle_{\mathrm{nil}}\times G_\Box \longrightarrow A_1(G_\Box),$$

see Definition 3.5.

The laws of a stable quadratic module show that $P(-|-)_H$ is a central homomorphism which factors through the reduced tensor square

$$P(-|-)_H\colon \otimes^2\mathbb{Z}[G_+] \twoheadrightarrow \hat{\otimes}^2\mathbb{Z}[G_+] \longrightarrow A_1(G_\Box).$$

Equation (4) in Definition 3.5 for $s=t=1\in G_\Box$ shows that $\langle x,1\rangle = 0$ is always zero. Moreover, given arbitrary $s,t\in G_\Box$

$$\begin{aligned}\langle 1, st\rangle &= \langle \delta(s),t\rangle + \langle \varepsilon(t),s\rangle\\ &= \langle \delta(s),t\rangle + \varepsilon(t)^*\langle 1,s\rangle + \binom{\varepsilon(s)}{2}\binom{\varepsilon(t)}{2}P(1|1)_H.\end{aligned}$$

Here we also use Lemma 1.18. Since δ is surjective this equation shows that $\langle -,-\rangle$ is fully determined by $P(-|-)_H$ and the elements

$$\langle 1,t\rangle,\quad t\in G_\Box.$$

The group $A_1(G_\Box)$ has necessarily nilpotency class 2, so there is a well-defined homomorphism

$$\langle G_\Box\rangle_{\mathrm{nil}} \longrightarrow A_1(G_\Box)\colon t\mapsto \langle 1,t\rangle,$$

where $G_\Box$ is based at $1\in G_\Box$.

There is no further source of elements in $A_1(G_\Box)$, therefore $A_1(G_\Box)$ must be a quotient of

$$\langle G_\Box\rangle_{\mathrm{nil}}\times \hat{\otimes}^2\mathbb{Z}[G_+].$$

Taking $s=\omega\in G_\Box$ in the equation above and using (5) in Definition 3.5 we obtain for any $t\in G_\Box$ the relation

$$\langle 1,\omega t\rangle = \langle 1,t\rangle + P(1|1)_H. \tag{a}$$

By Definition 3.5 (2) and by the elementary laws of a quadratic pair module the structure homomorphisms P, ∂ are given by

$$\begin{aligned} P(h\otimes g) &= (0, g\hat{\otimes}h),\quad g,h\in G,\\ \partial(t,g\hat{\otimes}h) &= (-\delta(t)+\varepsilon(t)) + (-g-h+g+h),\quad t\in G_\Box.\end{aligned}$$

Then the following relation in $A_1(G_\Box)$ must be satisfied,

$$(-s-t+s+t,0) = (0,(-\delta(s)+\varepsilon(s))\hat{\otimes}(-\delta(t)+\varepsilon(t))),\quad s,t\in G_\Box. \tag{b}$$

From (a) we obtain a further relation in $A_1(G_\square)$,

$$(\omega t, 0) \quad = \quad (t,0) + (0, 1\hat{\otimes}1), \quad t \in G_\square. \tag{c}$$

A straightforward computation shows that if we define $A_1(G_\square)$ as the quotient of $\langle G_\square\rangle_{\mathrm{nil}} \times \hat{\otimes}^2\mathbb{Z}[G_+]$ by relations (b) and (c) then $A(G_\square)$ is indeed a well-defined 0-free quadratic pair module endowed with an action of $G_\square$, and with homology

$$\begin{aligned} h_0A(G_\square) &\cong \mathbb{Z}, \quad \text{with natural projection } \langle G_+\rangle_{\mathrm{nil}} \twoheadrightarrow \mathbb{Z}\colon g \mapsto \varepsilon(g), \\ h_1A(G_\square) &\cong \mathbb{Z}/2, \quad \text{generated by } (\omega, 0) = (0, 1\hat{\otimes}1), \end{aligned}$$

so ξ is a weak equivalence. Notice that the action of G on $A_{ee}(G_\square)$ is the diagonal action and then the action of G on $A_1(G_\square)$ is determined by equation (3) in Definition 3.5. This finally defined a quadratic pair module $A(G_\square)$ and a morphism $\xi\colon \overline{\mathbb{Z}}_{\mathrm{nil}} \to A(G_\square)$ satisfying all the required properties.

For $G_\square = \mathrm{Sym}_\square(n)$ the weak equivalence $\overline{\mathbb{Z}}_{\mathrm{nil}} \to \Pi_{n,*}S^n$ given by (2.9) factors in a unique way as

$$\overline{\mathbb{Z}}_{\mathrm{nil}} \stackrel{\xi}{\longrightarrow} A(\mathrm{Sym}_\square(n)) \stackrel{\zeta}{\longrightarrow} \Pi_{n,*}S^n,$$

by a $\mathrm{Sym}_\square(n)$-equivariant morphism ζ of stable quadratic modules, i.e. ζ is given levelwise by $\mathrm{Sym}(n)$-equivariant group homomorphisms ζ_i, $i \in \{0, 1, ee\}$, and for any $x \in A_0(\mathrm{Sym}_\square(n))$ and $s \in \mathrm{Sym}_\square(n)$ the equation $\zeta_1\langle x, s\rangle = \langle \zeta_0(x), s\rangle$ holds. This follows from the definition of $\mathrm{Sym}_\square(n)$ in terms of tracks between maps $S^n \to S^n$, as $\Pi_{n,1}S^n$. The quadratic pair module morphism ζ is then a weak equivalence, since weak equivalences of quadratic pair modules clearly satisfy the "two out of three" property.

6 The structure of the symmetric track groups

In this section we construct a positive pin representation for the symmetric track group $\mathrm{Sym}_\square(n)$. By using this representation we obtain a finite presentation of $\mathrm{Sym}_\square(n)$.

The action of $\mathrm{Sym}(n)$ on S^n can be extended to a well-known action of the orthogonal group $O(n)$ which we now recall. Let $[-1,1]^n \subset \mathbb{R}^n$ be the hypercube centered in the origin whose vertices have all coordinates in $\{\pm 1\}$, $D^n \subset \mathbb{R}^n$ the Euclidean unit ball and S^{n-1} its boundary. There is a homeomorphism $\phi\colon [-1,1]^n \to D^n$ fixing the origin defined as follows

$$\phi(\underline{x}) = \frac{\max\limits_{1\leq i\leq n} |x_i|}{\|\underline{x}\|}\,\underline{x}.$$

Here $\underline{x} \in [-1,1]^n$ is an arbitrary non-trivial vector in the hypercube and $\|\cdot\|$ is the Euclidean norm. This homeomorphism projects the hypercube onto the ball from the origin. There is also a map collapsing the boundary

$$\varrho\colon [-1,1]^n \longrightarrow S^1 \wedge \stackrel{n}{\cdots} \wedge S^1 = S^n,$$

$$\varrho(x_1, \dots, x_n) = (\exp(i\pi(1+x_1)), \dots, \exp(i\pi(1+x_n))).$$

The composite

$$\varrho\phi^{-1}\colon D^n \longrightarrow S^n$$

induces a homeomorphism

$$D^n/S^{n-1} \cong S^1 \wedge \overset{n}{\cdots} \wedge S^1 = S^n$$

that we fix.

The orthogonal group $O(n)$ acts on the left of the unit ball D^n. This action induces an action of $O(n)$ on the quotient space $S^n = D^n/S^{n-1}$ preserving the base-point. The interchange of coordinates action of the symmetric group $\mathrm{Sym}(n)$ on $\mathbb{R}^n$ preserves the Euclidean scalar product, and therefore induces a homomorphism

$$i\colon \mathrm{Sym}(n) \hookrightarrow O(n). \tag{6.1}$$

The pull-back of the action of $O(n)$ along this homomorphism is the action of $\mathrm{Sym}(n)$ on S^n given by the smash product decomposition of S^n.

Remark 6.2. The action of $O(n)$ on S^n defines an inclusion $O(n) \subset \mathrm{End}_*(S^n)$. The induced homomorphism on π_1 is the Whitehead-Hopf J-homomorphism

$$J\colon \pi_1 O(n) \cong \pi_1 \mathrm{End}_*(S^n) = \pi_{n+1} S^n \tag{6.3}$$

which is known to be an isomorphism for $n \geq 2$. Let $\pi_{0,*}O(n)$ be the fundamental groupoid of the Lie group $O(n)$. Then, considering elements $A, B \in O(n)$ as pointed maps

$$A, B\colon S^n \longrightarrow S^n$$

the isomorphism in (6.3) allows to identify all morphisms $\gamma\colon A \to B$ in $\pi_{0,*}O(n)$ with all tracks

$$\gamma\colon A \Rightarrow B$$

in $\pi_{0,1} \mathrm{End}_*(S^n)$. Let $\mathrm{Id}_n \in O(n)$ be the identity matrix. The order 2 matrix

$$\begin{pmatrix} \mathrm{Id}_{n-1} & 0 \\ 0 & -1 \end{pmatrix} \in O(n)$$

will be denoted by $\mathrm{Id}_{n-1} \oplus (-1)$. By using the action of $O(n)$ on S^n we have by the notation in (4.3) that

$$\mathrm{Id}_{n-1} \oplus (-1) = (\cdot)_n^{-1}\colon S^n \longrightarrow S^n.$$

Obviously $\mathrm{Id}_n = (\cdot)_n^1 = 1_{S^n}\colon S^n \to S^n$.

The topological group structure of $O(n)$ induces an internal group structure on the fundamental groupoid $\pi_{0,*}O(n)$ in the category of groupoids. In particular the set $\pi_{0,1}O(n)$ of morphisms in $\pi_{0,*}O(n)$ forms a group. We define the subgroup

$$\widetilde{O}(n) \subset \pi_{0,1}O(n)$$

consisting of all the morphisms with target Id_n or $\mathrm{Id}_{n-1} \oplus (-1)$. By Remark 6.2 the symmetric track group is the subgroup

$$\mathrm{Sym}_{\square}(n) \subset \widetilde{O}(n)$$

of morphisms with source in the image of i in (6.1), $n \geq 3$. The subgroup $\widetilde{O}(n)$ is embedded in an extension

$$\mathbb{Z}/2 \hookrightarrow \widetilde{O}(n) \stackrel{q}{\twoheadrightarrow} O(n), \quad n \geq 3. \tag{6.4}$$

The projection q sends a morphism in $\widetilde{O}(n) \subset \pi_{0,1}O(n)$ to the source, and the kernel is clearly $\pi_1 O(n) = \mathbb{Z}/2$ for $n \geq 3$. The case $n = 2$ will be considered in Remark 6.12 below.

There is also a well-known extension

$$\mathbb{Z}/2 \hookrightarrow Pin_+(n) \stackrel{\rho}{\twoheadrightarrow} O(n) \tag{6.5}$$

given by the positive pin group. Let us recall the definition of this extension.

Definition 6.6. The positive Clifford algebra $C_+(n)$ is the unital $\mathbb{R}$-algebra generated by e_i, $1 \leq i \leq n$, with relations

1. $e_i^2 = 1$ for $1 \leq i \leq n$,
2. $e_i e_j = -e_j e_i$ for $1 \leq i < j \leq n$.

Clifford algebras are defined for arbitrary quadratic forms on finite-dimensional vector spaces, see for instance [BtD85, 6.1]. The Clifford algebra defined above corresponds to the quadratic form of the standard positive-definite scalar product in $\mathbb{R}^n$. We identify the sphere S^{n-1} with the vectors of Euclidean norm 1 in the vector subspace $\mathbb{R}^n \subset C_+(n)$ spanned by the generators e_i. The vectors in S^{n-1} are units in $C_+(n)$. Indeed for any $v \in S^{n-1}$ the square $v^2 = 1$ is the unit element in $C_+(n)$, so that $v^{-1} = v$. The group $Pin_+(n)$ is the subgroup of units in $C_+(n)$ generated by S^{n-1}. Any $x \in Pin_+(n)$ defines an automorphism of $\mathbb{R}^n \subset C_+(n)$ given by conjugation in $C_+(n)$ as follows

$$\mathbb{R}^n \longrightarrow \mathbb{R}^n \colon w \mapsto -xwx^{-1}.$$

If $x \in S^{n-1}$ then this automorphism is the reflection along the hyperplane orthogonal to the unit vector x. This endomorphism always preserves the scalar product, therefore this defines a homomorphism

$$\rho\colon Pin_+(n) \twoheadrightarrow O(n).$$

This homomorphism is surjective since all elements in $O(n)$ are products of $\leq n$ reflections. It is easy to see that the kernel of ρ is $\mathbb{Z}/2$ generated by $-1 \in C_+(n)$. This is the extension in (6.5).

The Clifford algebra $C_+(n)$ has dimension 2^n. A basis is given by the elements

$$e_{i_1} \cdots e_{i_k}, \;\; 1 \leq i_1 < \cdots < i_k \leq n.$$

We give $C_+(n)$ the topology induced by the Euclidean norm associated to this basis. The positive pin group inherits a topology turning (6.5) into a Lie group extension.

Proposition 6.7. *The extension (6.4) is isomorphic to (6.5).*

Proof. Since $Pin_+(n)$ is a topological group $\pi_{0,*}Pin_+(n)$ is a group object in the category of groupoids. We define

$$\widetilde{Pin}_+(n) \subset \pi_{0,1}Pin_+(n)$$

to be the subgroup given by morphisms $x \to y$ in $\pi_{0,*}Pin_+(n)$ with target 1 or e_n. This is well defined since $\{1, e_n\} \subset Pin_+(n)$ is a subgroup. This observation is indeed the key step of the proof, and it shows for example why the negative pin group does not occur as (6.4). Moreover, $\pi_{0,*}\rho$ induces a homomorphism

$$\widetilde{Pin}_+(n) \longrightarrow \widetilde{O}(n). \tag{a}$$

It is well-known that $Pin_+(n)$ has two components. The two components are separated by the function

$$Pin_+(n) \overset{\rho}{\twoheadrightarrow} O(n) \overset{\det}{\twoheadrightarrow} \{\pm 1\}\,.$$

In particular 1 and e_n lie in different components, hence the homomorphism

$$\widetilde{Pin}_+(n) \longrightarrow Pin_+(n) \tag{b}$$

is surjective. Moreover, it is injective since the two components of $Pin_+(n)$ are known to be simply connected, therefore (b) is an isomorphism. The inverse

$$Pin_+(n) \longrightarrow \widetilde{Pin}_+(n) \tag{c}$$

sends an element $x \in Pin_+(n)$ to the image by $\pi_{0,*}\rho$ of the unique morphism $x \to y$ in $\pi_{0,*}Pin_+(n)$ with $y = e_n$ provided $\det \rho(x) = -1$ or $y = 1$ otherwise.

Obviously the composite of (c) and (a) is compatible with the projections onto $O(n)$ in (6.4) and (6.5), so we only need to check that the composite of (c) and (a) induces an isomorphism between the kernels. The kernel of ρ is -1. A path $\gamma\colon [0,1] \to Pin_+(n)$ from -1 to 1 is defined by

$$\gamma(t) = (-\cos(t\pi)e_2 + \sin(t\pi)e_1)e_2 = -\cos(t\pi) + \sin(t\pi)e_1e_2.$$

Now it is an easy exercise to check that $\rho\gamma\colon [0,1] \to O(n)$ is a generator of $\pi_1 O(n)$, and hence we are done. ∎

Remark 6.8. We recall that the extension (6.5), and therefore (6.4), represents the second Stiefel-Whitney class $w_2 \in H^2(BO(n), \mathbb{Z}/2)$, compare [Tei92, page 21].

By definition of (6.4) and Proposition 6.7 we obtain the following corollary.

Corollary 6.9. *For $n \geq 3$ the symmetric track group $\mathrm{Sym}_\square(n)$ is the pull back of the central extension for the positive pin group $Pin_+(n)$ in (6.5) along the inclusion $i\colon \mathrm{Sym}(n) \subset O(n)$, in particular there is a central extension*

$$\mathbb{Z}/2 \hookrightarrow \mathrm{Sym}_\square(n) \overset{\delta}{\twoheadrightarrow} \mathrm{Sym}(n)$$

classified by the pull-back of the second Stiefel-Whitney class $i^*w_2 \in H^2(\mathrm{Sym}(n), \mathbb{Z}/2)$.

Remark 6.10. The low-dimensional mod 2 cohomology groups of symmetric groups Sym(n) are as follows, $n \geq 3$,

$$H^1(\mathrm{Sym}(n), \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2\,\chi \oplus \mathbb{Z}/2\,i^*w_1, & \text{for } n = 3; \\ \mathbb{Z}/2\,i^*w_1, & \text{for } n > 3; \end{cases}$$

$$H^2(\mathrm{Sym}(n), \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2\,i^*w_1^2, & \text{for } n = 3; \\ \mathbb{Z}/2\,i^*w_1^2 \oplus \mathbb{Z}/2\,i^*w_2, & \text{for } n > 3. \end{cases}$$

Here we write $w_j \in H^j(BO(n), \mathbb{Z}/2)$ for the j^{th} Stiefel-Whitney class, $j = 1, 2$. The pull-back i^*w_1 corresponds to the sign homomorphism

$$i^*w_1 = \mathrm{sign}\colon \mathrm{Sym}(n) \longrightarrow \{\pm 1\} \cong \mathbb{Z}/2,$$

The pull-back of the second Stiefel-Whitney class is trivial for $n = 3$, therefore $\mathrm{Sym}_\square(3)$ is a split extension of Sym(3) by $\mathbb{Z}/2$, and $\chi\colon \mathrm{Sym}_\square(3) \twoheadrightarrow \mathbb{Z}/2$ is a retraction.

The following structure theorem follows from Corollary 6.9.

Theorem 6.11. *The symmetric track group* $\mathrm{Sym}_\square(n)$ *is the subgroup of* $Pin_+(n)$ *formed by the units* $x \in C_+(n)$ *such that for any* $1 \leq i \leq n$ *there exists* $1 \leq \sigma(i) \leq n$ *with* $-xe_ix^{-1} = e_{\sigma(i)}$. *The boundary homomorphism* $\delta\colon \mathrm{Sym}_\square(n) \twoheadrightarrow \mathrm{Sym}(n)$ *sends* x *above to the permutation* $\delta(x) = \sigma$. *The group* $\mathrm{Sym}_\square(n)$ *has a presentation given by generators* ω, t_i, $1 \leq i \leq n-1$, *and relations*

$$\begin{aligned} t_1^2 &= 1 \text{ for } 1 \leq i \leq n-1, \\ (t_it_{i+1})^3 &= 1 \text{ for } 1 \leq i \leq n-2, \\ \omega^2 &= 1, \\ t_i\omega &= \omega t_i \text{ for } 1 \leq i \leq n-1, \\ t_it_j &= \omega t_jt_i \text{ for } 1 \leq i < j-1 \leq n-1; \end{aligned}$$

with $\omega \mapsto -1$ *and* $t_i \mapsto \frac{1}{\sqrt{2}}(e_i - e_{i+1})$. *In particular* $\delta(\omega) = 0$ *and* $\delta(t_i) = (i\; i+1)$.

This is a group considered by Schur in [Sch11] and by Serre in [Ser84].

Remark 6.12. In case $n = 2$ we have $O(2) = \{\pm 1\} \ltimes S^1$ with $\{\pm 1\}$ acting on S^1 exponentially, $\widetilde{O}(2) = \{\pm 1\} \ltimes \mathbb{R}$ with $\{\pm 1\}$ acting on $\mathbb{R}$ multiplicatively, and the projection $q\colon \widetilde{O}(2) \twoheadrightarrow O(2)$ defined as in (6.4) is the identity in $\{\pm 1\}$ and the exponential map in the second coordinate $\mathbb{R} \twoheadrightarrow S^1\colon x \mapsto \exp(2\pi i x)$. In particular we have an abelian extension

$$\mathbb{Z} \hookrightarrow \widetilde{O}(2) \overset{q}{\twoheadrightarrow} O(2).$$

The induced action of $O(2)$ on $\mathbb{Z}$ is given by the determinant $\det\colon O(2) \twoheadrightarrow \{\pm 1\}$. By Remark 6.2 the extended symmetric track group $\overline{\mathrm{Sym}_\square}(2)$ is the pull-back of $i\colon \mathrm{Sym}(2) \subset O(2)$ along q, therefore we have an abelian extension

$$\mathbb{Z} \hookrightarrow \overline{\mathrm{Sym}_\square}(2) \overset{q}{\twoheadrightarrow} \mathrm{Sym}(2), \tag{6.13}$$

where $\mathrm{Sym}(2)$ acts on $\mathbb{Z}$ by the unique isomorphism $\mathrm{Sym}(2) \cong \{\pm 1\}$. Now the symmetric track group $\mathrm{Sym}_\square(2)$ can be identified with the push-forward of the extension (6.13) along the natural projection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$, therefore we get a central extension

$$\mathbb{Z}/2 \hookrightarrow \mathrm{Sym}_\square(2) \overset{q}{\twoheadrightarrow} \mathrm{Sym}(2). \tag{6.14}$$

The cohomology group $H^2(\mathrm{Sym}(2), \mathbb{Z}) = 0$ is trivial, so (6.13) is a split extension and $\overline{\mathrm{Sym}}_\square(2) \cong \mathrm{Sym}(2) \ltimes \mathbb{Z}$ is a semidirect product. Moreover (6.14) is also split since it is the push-forward of (6.13), therefore $\mathrm{Sym}_\square(2) \cong \mathrm{Sym}(2) \times \mathbb{Z}/2$ is a product.

7 An application to the cup-one product

Let $n \geq m > 1$ be even integers. The cup-one product operation

$$\pi_n S^m \longrightarrow \pi_{2n+1} S^{2m} \colon \alpha \mapsto \alpha \smile_1 \alpha$$

is defined in the following way, compare [HM93, 2.2.1]. Let k be any positive integer and let $\tau_k \in \mathrm{Sym}(2k)$ be the permutation exchanging the first and the second block of k elements in $\{1, \dots, 2k\}$. If k is even then $\operatorname{sign} \tau_k = 1$. We choose for any even integer $k > 1$ a track $\hat{\tau}_k \colon \tau_k \Rightarrow 1_{S^{2k}}$ in $\mathrm{Sym}_\square(2k)$. Consider the following diagram in the track category $\mathbf{Top}^*$ of pointed spaces where $a \colon S^n \to S^m$ represents α.

$$\begin{array}{ccccc} & S^{2n} & \xrightarrow{a \wedge a} & S^{2m} & \\ 1_{S^{2n}} \; \overset{\hat{\tau}_n^{\boxminus}}{\Longrightarrow} & \downarrow{\scriptstyle \tau_n} & & {\scriptstyle \tau_m}\downarrow & \overset{\hat{\tau}_m}{\Longrightarrow} \; 1_{S^{2m}} \\ & S^{2n} & \xrightarrow{a \wedge a} & S^{2m} & \end{array} \tag{7.1}$$

By pasting this diagram we obtain a self-track of $a \wedge a$

$$(\hat{\tau}_m(a \wedge a)) \square ((a \wedge a)\hat{\tau}_n^{\boxminus}) \colon a \wedge a \Rightarrow a \wedge a. \tag{7.2}$$

The set of self-tracks $a \wedge a \Rightarrow a \wedge a$ is the automorphism group of the map $a \wedge a$ in the track category $\mathbf{Top}^*$. The element $\alpha \smile_1 \alpha \in \pi_{2n+1} S^{2m}$ is given by the track (7.2) via the well-known Barcus-Barratt-Rutter isomorphism

$$\mathrm{Aut}(a \wedge a) \cong \pi_{2n+1} S^{2m},$$

see [BB58], [Rut67] and also [Bau91, VI.3.12] and [BJ01] for further details.

The following theorem generalizes [BJM83, 6.5].

Theorem 7.3. *The formula*

$$2(\alpha \smile_1 \alpha) = \frac{n+m}{2} (\alpha \wedge \alpha)(\Sigma^{2(n-1)} \eta)$$

holds, where $\eta \colon S^3 \to S^2$ is the Hopf map.

The proof of Theorem 7.3 is based on the following lemma.

Lemma 7.4. *The following formula holds in* $\mathrm{Sym}_\square(2k)$

$$\hat{\tau}_k^2 = \omega^{\binom{k}{2}}.$$

Proof. Here we use the representation of $\mathrm{Sym}_\Box(2k)$ in $Pin_+(2k)$ given in Theorem 6.11 and the relations (1) and (2) in the definition of the Clifford algebra $C_+(2k)$, see Definition 6.6.

The permutation τ_k can be expressed as a product of transpositions as follows

$$\tau_k = (1\ \ k)(2\ \ k+1)\cdots(k-1\ \ 2k-1)(k\ \ 2k).$$

The element $\frac{1}{\sqrt{2}}(e_i - e_{i+k}) \in S^{2k-1} \subset Pin_+(2k)$ acts on $\mathbb{R}^{2k}$ (with coordinates e_i, $1 \leq i \leq 2k$) by reflection along the hyperplane orthogonal to $\frac{1}{\sqrt{2}}(e_i - e_{i+k})$, see Definition 6.6. This hyperplane is $e_i = e_{i+k}$, therefore the action of $\frac{1}{\sqrt{2}}(e_i - e_{i+k})$ on $\mathbb{R}^{2k}$ interchanges the coordinates in e_i and e_{i+k} and preserves all the other ones.

Now by Theorem 6.11 $\frac{1}{\sqrt{2}}(e_i - e_{i+k})$ lies in the positive pin representation of $\mathrm{Sym}_\Box(2k)$ and $\delta\left(\frac{1}{\sqrt{2}}(e_i - e_{i+k})\right) = (i\ \ i+k)$, so

$$\hat{\tau}_k = \pm\frac{1}{2^{\frac{k}{2}}}(e_1 - e_{k+1})(e_2 - e_{k+2})\cdots(e_{k-1} - e_{2k-1})(e_k - e_{2k}).$$

The following equalities hold in the Clifford algebra $C_+(2k)$, see the defining relations in Definition 6.6, $i \neq j$, $i \neq j+k$, $i+k \neq j$, $k > 0$,

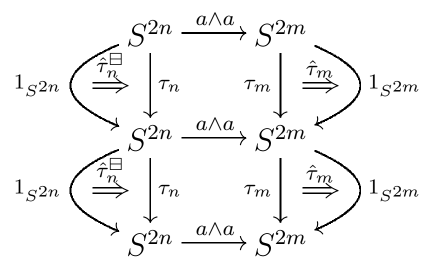
$$\begin{aligned}
(e_i - e_{i+k})^2 &= e_i^2 - e_ie_{i+k} - e_{i+k}e_i + e_{i+k}^2\\
&= 1 - e_ie_{i+k} + e_ie_{i+k} + 1\\
&= 2,\\
(e_i - e_{i+k})(e_j - e_{j+k}) &= e_ie_j - e_ie_{j+k} - e_{i+k}e_j + e_{i+k}e_{j+k}\\
&= -e_je_i + e_{j+k}e_i + e_je_{i+k} - e_{j+k}e_{i+k}\\
&= -(e_j - e_{j+k})(e_i - e_{i+k}).
\end{aligned}$$

Hence we observe that

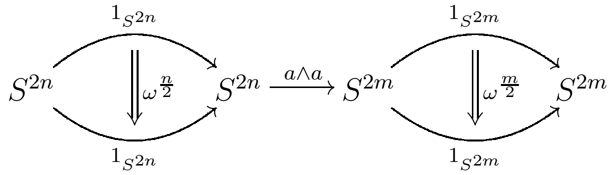
$$\begin{aligned}
\hat{\tau}_k^2 &= \frac{1}{2^{\frac{k}{2}}}\frac{1}{2^{\frac{k}{2}}}(-1)^{k-1}2(-1)^{k-2}2\cdots(-1)^1 2(-1)^0 2\\
&= (-1)^{k-1}(-1)^{k-2}\cdots(-1)^1(-1)^0\\
&= (-1)^{\binom{k}{2}}.
\end{aligned}$$

The proof is now finished. ■

Proof of Theorem 7.3. The element $2(\alpha \smile_1 \alpha)$ corresponds to the pasting of the following diagram



By Lemma 7.4 and using that n and m are even this composite track coincides with



therefore $2(\alpha \smile_1 \alpha)$ corresponds to the self-track

$$((\omega^{\frac{m}{2}})(a \wedge a))\square((a \wedge a)(\omega^{\frac{n}{2}})). \tag{a}$$

The self-track $\omega^{\frac{m}{2}}(a \wedge a)$ corresponds to the homotopy class

$$\left(\frac{m}{2}(\Sigma^{2(m-1)}\eta)\right)(\Sigma(\alpha \wedge \alpha)). \tag{b}$$

Since $\Sigma(\alpha \wedge \alpha) = \pm(\Sigma^{m+1}\alpha)(\Sigma^{n+1}\alpha)$ which is a composite of two triple suspensions (b) is

$$(\alpha \wedge \alpha)\left(\frac{m}{2}(\Sigma^{2(n-1)}\eta)\right). \tag{c}$$

Moreover, the self-track $(a \wedge a)(\omega^{\frac{n}{2}})$ corresponds to

$$(\alpha \wedge \alpha)\left(\frac{n}{2}(\Sigma^{2(n-1)}\eta)\right), \tag{d}$$

so the self-track (a) corresponds to the sum of (c) and (d)

$$(\alpha \wedge \alpha)\left(\frac{n+m}{2}(\Sigma^{2(n-1)}\eta)\right). \qquad \blacksquare$$

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