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Representation Theory of some Infinite-dimensional Algebras Arising in Continuously Controlled Algebra and Topology

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Abstract. In this paper we determine the representation type of some algebras of infinite matrices continuously controlled at infinity by a compact metrizable space. We explicitly classify their finitely presented modules in the finite and tame cases. The algebra of row-column-finite (or locally finite) matrices over an arbitrary field is one of the algebras considered in this paper, its representation type is shown to be finite.

Key words: representation theory, controlled algebra, infinite matrices.

1. Introduction

Suppose that the discrete set $\mathbb{N}_0$ of non-negative integers is embedded $\mathbb{N}_0 \subset X$ in a compact metrizable space X, and let $E = \mathbb{N}'_0 \subset X$ be the derived set (i.e. points of X which contain infinitely many points of $\mathbb{N}_0$ in any neighborhood). Consider the set $R(E)$ of $\mathbb{N}_0 \times \mathbb{N}_0$ matrices $(\mathsf{a}_{ij})_{i,j\in\mathbb{N}_0}$ with entries in a (unital associative) ring R such that if $\{i_n\}_{n\in\mathbb{N}_0}$, $\{j_n\}_{n\in\mathbb{N}_0} \subset \mathbb{N}_0$ are sequences convergent in X to different points then the vector $(\mathsf{a}_{i_n j_n})_{n\in\mathbb{N}_0}$ is almost all zero. This set is an R-algebra with the usual matrix operations. Any compact metrizable space can arise as E in this way. In fact the isomorphism class of the algebra $R(E)$ only depends on E. These algebras are Morita equivalent to some additive categories of free R-modules continuously controlled at infinity by E appearing in the literature. These categories play an important role in many areas such as controlled homotopy theory, proper homotopy theory, C^*-algebra theory, K-theory and L-theory (see for example [1,2,5,9,16]).

The elementary properties of the algebras $R(E)$ have been studied by Baues–Quintero ([2], V.3) for $R = \mathbb{Z}$ the integers. If E is zero-dimensional, $R(E)$ is a particular case of the rings considered by Farrell–Wagoner [7]. When $E = *$ is a singleton $R(E) = \mathrm{RCFM}(R)$ is the well-known algebra of row-column-finite (or locally finite) matrices over R. This algebra has been studied from a purely ring-theoretical point of view (see for example [18]). It was also used by Wagoner [19] to construct deloopings in algebraic K-theory.

In this paper we concentrate on the representation theory of the algebras $R(E)$. Representation theory considers the *decomposition problem* in a small additive category $\mathbf{A}$. A solution to this problem consists of a set of objects (which we call *elementary objects*) and of a set of isomorphisms (*elementary isomorphisms*) between finite direct sums of elementary objects. These sets must satisfy the following properties: any object in $\mathbf{A}$ is isomorphic to a finite direct sum of elementary ones, and any isomorphism relation between two such direct sums can be derived from the elementary isomorphisms. Notice that this is exactly a presentation of the abelian monoid $\operatorname{Iso}(\mathbf{A})$ of isomorphisms classes of objects in $\mathbf{A}$. The trivial solution is taking all objects as elementary objects and all isomorphisms as elementary isomorphisms, however one is often interested in solutions minimizing the cardinal of the sets of elementary objects and isomorphisms.

We say that $\mathbf{A}$ has *finite representation type* if there exists a finite set of elementary objects and isomorphisms, or equivalently $\operatorname{Iso}(\mathbf{A})$ is finitely presented. The representation type of $\mathbf{A}$ is *wild* if a solution to the decomposition problem in $\mathbf{A}$ would yield a solution to the decomposition problem in the category of finite-dimensional modules over a polynomial k-algebra in two non-commuting variables. Otherwise $\mathbf{A}$ has *tame* representation type. If $\mathbf{A}$ has wild representation type the word problem for finitely presented groups, which is undecidable, can be embedded in the decomposition problem in $\mathbf{A}$, hence one can not expect to get satisfactory solutions in this case. The representation type of an algebra A is that of the category $\mathbf{fp}(A)$ of finitely presented (right) A-modules.

We prove in Corollary 4.4 that the decomposition problem for finitely presented $R(E)$-modules contains the decomposition problem for countably presented R-modules. This makes this problem untractable even for such an elementary ring as $R=\mathbb{Z}$ since all countable abelian groups are countably presented and conversely. For this reason, in the most of the paper we restrict ourselves to the case $R=k$ an arbitrary field.

One of the main results of this paper is the following theorem, where we compute the representation type of the algebra $k(E)$ in terms of the cardinality of E, without restrictions on the ground field k.

THEOREM 1.1. *The representation type of* $k(E)$ *is*

card E	type
<4	*finite*
$=4$	*tame*
>4	*wild*

In the finite and tame cases we construct explicit presentations of $\mathrm{Iso}(\mathbf{fp}(k(E)))$. Moreover, for E finite, we prove that there are presentations of $\mathrm{Iso}(\mathbf{fp}(k(E)))$ with a finite number of elementary isomorphisms and we compute them. These presentations satisfy the following properties.

THEOREM 1.2. *If* card E *is finite there are solutions to the decomposition problem in* $\mathbf{fp}(k(E))$ *with the following cardinals of elementary modules and isomorphisms*

card E	modules	isomorphisms
1	6	6
2	12	12
3	21	18
$\geqslant 4$	$\geqslant \aleph_0$	6 card E

There are two key steps in the proof of these results. The first is to solve the decomposition problem for finitely presented $\mathrm{RCFM}(k)$-modules. The second is to relate the decomposition problem in $\mathbf{fp}(k(E))$ when card $E=n$ is finite to the decomposition problems in $\mathbf{fp}(\mathrm{RCFM}(k))$ and in the category of finite-dimensional n-subspaces.

We shall use the category of pro-vector spaces and the inverse limit functor to construct invariants detecting isomorphism types of finitely presented $k(E)$-modules. Moreover, we shall not usually work with the categories of finitely presented $k(E)$-modules but with the equivalent categories of finitely presented modules over certain small additive categories. This alternative setting allows more flexibility and technical proofs become less complicated than if we use $k(E)$-modules.

The results of this paper will be applied to proper homotopy theory in a forthcoming paper [14]. Homotopy and homology 'groups' in the proper homotopy category of spaces with a fixed space of Freudenthal ends E are in fact $\mathbb{Z}(E)$-modules. Proper cohomology groups are honest abelian groups but their coefficients are $\mathbb{Z}(E)$-modules. In proper obstruction theory we find some relevant cohomology groups with coefficients in a proper homology module tensored by $\mathbb{Z}/2$. These coefficient modules are hence $\mathbb{F}_2(E)$-modules, where $\mathbb{F}_2$ is the field with two elements. The representation theory developed in this paper plays a crucial role in the computation of some classes in cohomology of categories coming from proper obstruction theory. These computations lead us to determine in [14] algebraic models for the proper homotopy types of a certain class of spaces already considered by Whitehead in ordinary homotopy theory [20].

In order to ease the reading we now describe the contents of this paper. In Section 2 we briefly recall from [13] the basic tools of ringoid theory that we need. Afterwards, in Section 3, we introduce the ringoids which are Morita equivalent to the algebras $R(E)$ and establish their basic properties. For this we use the approach in [2], generalizing some results in this reference for $R=\mathbb{Z}$ to arbitrary rings. We put emphasis on the case E finite because we shall always work under this assumption (even for the proof of Theorem 1.1, see Remark 3.10). In Section 4 we construct an embedding of the decomposition problem for countably presented R-modules into the decomposition problem for finitely presented $R(E)$-modules, where E is any non-empty compact metrizable space. Section 5 contains basic facts about pro-categories. In Section 6 we construct some invariants of the isomorphism class of a finitely presented $k(E)$-module, E finite. These invariants are used in Section 7 to classify finitely presented $k(E)$-modules when $E=*$ is just one point and $k(E)=\mathrm{RCFM}(k)$. In particular we prove that this k-algebra has finite representation type. The classification theorem (Theorem 7.1) is derived from several technical lemmas. In Section 8 we recall the definition and representation theory of the n-subspace quiver. We also define the class of rigid n-subspaces, which plays an important role in what follows. We show that all but $3n$ indecomposable representations of the n-subspace quiver are rigid n-subspaces. In Section 9 we relate the representation theories of both $k(E)$ and the n-subspace quiver, where n is the cardinality of E. The properties of this relation are established in a series of technical results which lead us to complete the proofs of Theorems 1.1 and 1.2 in Section 10. In this last section we compute the structure of the monoid $\mathrm{Iso}(\mathbf{fp}(k(E)))$ (Theorem 10.1) and give a classification theorem for finitely presented $k(E)$-modules (Corollary 10.5) for E finite. This classification theorem explicitly describes the (finite) set of elementary isomorphisms, and also the set of elementary objects when E has less than 5 points. We include an Appendix with some computations of Ext^1 groups of finitely presented $k(E)$-modules, E finite, which will be very useful for [14].

1.1. NOTATION AND CONVENTIONS

In this paper all rings and algebras are associative with unit. We use bold letters $\mathbf{C}$ for categories, R for an arbitrary (non-commutative) ring, $\mathbb{Z}$ for the ring of integers, and k for fields. As usual $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of natural numbers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the free abelian monoid with one generator.

Capital sans serif letters A are names of matrices with entries in some ring. Here all matrices are square matrices indexed by $\mathbb{N}_0$, and the entry of a matrix A corresponding to the subscripts i, $j \in \mathbb{N}_0$ is denoted by a_{ij}, that is $\mathsf{A} = (\mathsf{a}_{ij})_{i,j\in\mathbb{N}_0}$. The identity matrix is denoted by I, it is defined as $\mathsf{i}_{ii} = 1$ $(i \in \mathbb{N}_0)$ and $\mathsf{i}_{ij} = 0$ for $i \neq j$. The entries of the transposed matrix $\mathsf{A}^t = (\mathsf{a}^t_{ij})_{i,j\in\mathbb{N}_0}$ of A are $\mathsf{a}^t_{ij} = \mathsf{a}_{ji}$ $(i, j \in \mathbb{N}_0)$. Vectors are denoted by $(\mathsf{v}_i)_{i=1}^n$ or $(\mathsf{v}_n)_{n\in\mathbb{N}_0}$ provided they have a finite or an infinite countable number of entries. We regard vectors as column matrices, hence matrices act on vectors on the left.

2. Ringoids and Modules

In this section we recall basic facts about modules over a small ringoid. Our main reference for this subject is [13].

A *ringoid* $\mathbf{R}$ is a category whose morphism sets $\mathrm{Hom}_{\mathbf{R}}(X, Y)$ are abelian groups in such a way that composition is bilinear. The endomorphism set $\mathrm{End}_{\mathbf{R}}(X) = \mathrm{Hom}_{\mathbf{R}}(X, X)$ of an object X has a ring structure with product given by composition of morphisms. Conversely any ring R is identified with the ringoid with a single object whose endomorphism set is R. An *additive category* is a ringoid with finite biproducts (direct sums).

An *additive functor* between ringoids is a functor which induces homomorphisms between morphism sets. Let $\mathbf{Ab}$ be the category of abelian groups. A *right-*$\mathbf{R}$*-module* $\mathcal{M}$ is an additive functor $\mathcal{M}: \mathbf{R}^{op} \to \mathbf{Ab}$. Morphisms of right-$\mathbf{R}$-modules are natural transformations, and the category of right-$\mathbf{R}$-modules is denoted by $\mathbf{mod}(\mathbf{R})$ whenever $\mathbf{R}$ is small. Left-$\mathbf{R}$-modules are the same thing as right-$\mathbf{R}^{op}$-modules, where $\mathbf{R}^{op}$ is the opposite category, so every statement about right-modules has a convenient translation to left-modules. From now on every module is a right-module unless we state the contrary.

There is a Yoneda full inclusion of categories $\mathbf{R} \subset \mathbf{mod}(\mathbf{R})$ which sends an object X in $\mathbf{R}$ to the associated contravariant representable functor $\mathrm{Hom}_{\mathbf{R}}(-, X)$. These $\mathbf{R}$-modules are said to be *finitely generated free*. They are projective by Yoneda's lemma.

An $\mathbf{R}$-module $\mathcal{M}$ is *finitely presented* (f. p.) if it is the cokernel of a morphism between two finite direct sums of finitely generated free $\mathbf{R}$-modules. The cokernel of a morphism between f. p. modules is also f. p. In particular direct summands of f. p. modules are f. p. We write $\mathbf{fp}(\mathbf{R}) \subset \mathbf{mod}(\mathbf{R})$ for the full subcategory of f. p. $\mathbf{R}$-modules. If $\mathbf{R} = \mathbf{A}$ is an additive category, then a f. p. $\mathbf{A}$-module $\mathcal{M}$ is in fact the cokernel in $\mathbf{mod}(\mathbf{A})$ of a morphism

$\varphi: X_1 \to X_0$ in $\mathbf{A}$. One can readily check that if $\mathcal{N} = \mathrm{Coker}[\psi: Y_1 \to Y_0]$ is another f. p. $\mathbf{A}$-module, any morphism $\tau: \mathcal{M} \to \mathcal{N}$ is represented by a morphism $\tau_0: X_0 \to Y_0$ such that there exists $\tau_1: X_1 \to Y_1$ with $\tau_0\varphi = \psi\tau_1$. Another morphism $\tau_0': X_0 \to Y_0$ represents τ if and only if there exists $\eta: X_0 \to Y_1$ with $\tau_0 + \psi\eta = \tau_0'$. More precisely, let $\mathbf{pair}(\mathbf{A})$ be the additive category whose objects are morphisms in $\mathbf{A}$, and morphisms $\tau = (\tau_1, \tau_0): \varphi \to \psi$ are commutative squares.

$$\begin{tikzcd} X_1 \arrow[r, "\varphi"] \arrow[d, "\tau_1"'] & X_0 \arrow[d, "\tau_0"] \\ Y_1 \arrow[r, "\psi"] & Y_0 \end{tikzcd}$$

There is an obvious functor Coker: $\mathbf{pair}(\mathbf{A}) \to \mathbf{fp}(\mathbf{A})$ given by taking cokernels. We define in $\mathbf{pair}(\mathbf{A})$ the natural equivalence relation $\sim$ with $\tau \sim \tau'$ if there exists $\eta: X_0 \to Y_1$ satisfying $\tau_0 + \psi\eta = \tau_0'$.

PROPOSITION 2.1. *The functor* Coker *factors through the quotient category* $\mathbf{pair}(\mathbf{A})/\sim$ *and the induced functor* Coker: $\mathbf{pair}(\mathbf{A})/\sim \to \mathbf{fp}(\mathbf{A})$ *is an equivalence of categories.*

Any additive functor $\mathbb{F}: \mathbf{R} \to \mathbf{S}$ between small ringoids induces two 'change of coefficients' additive functors $\mathbb{F}^*: \mathbf{mod}(\mathbf{S}) \to \mathbf{mod}(\mathbf{R})$ and $\mathbb{F}_*: \mathbf{mod}(\mathbf{R}) \to \mathbf{mod}(\mathbf{S})$. The first one is exact and sends an $\mathbf{S}$-module $\mathcal{M}$ to the composite $\mathbb{F}^*\mathcal{M} = \mathcal{M}\mathbb{F}$. The second one is left-adjoint to $\mathbb{F}^*$ ($\mathbb{F}_*$ is the left additive Kan extension along $\mathbb{F}$, see [13], 6) and hence right-exact. Moreover, the next diagram commutes

$$\begin{tikzcd} \mathbf{R} \arrow[r, "\mathbb{F}"] \arrow[d, "\text{Yoneda}"'] & \mathbf{S} \arrow[d, "\text{Yoneda}"] \\ \mathbf{mod}(\mathbf{R}) \arrow[r, "\mathbb{F}_*"] & \mathbf{mod}(\mathbf{S}) \end{tikzcd} \tag{2.a}$$

In addition if $\mathbb{F}$ is full and faithful then so is $\mathbb{F}_*$, and in this case $\mathbb{F}^*\mathbb{F}_*$ is naturally equivalent to the identity, see [3], 3.4.1. This follows from the fact that any $\mathbf{R}$-module admits a projective resolution by (arbitrary) direct sums of finitely generated free ones. The functor $\mathbb{F}_*$ restricts to the full subcategories of f. p. modules.

If we identify the endomorphism ring $\mathrm{End}_{\mathbf{R}}(X)$ of an object X with the full subcategory of $\mathbf{R}$ whose unique object is X the change of coefficients $\mathbb{F}^*$ induced by the inclusion $\mathbb{F}: \mathrm{End}_{\mathbf{R}}(X) \subset \mathbf{R}$ is the evaluation functor

$$ev_X = \mathbb{F}^*: \mathbf{mod}(\mathbf{R}) \to \mathbf{mod}(\mathrm{End}_{\mathbf{R}}(X)): \mathcal{M} \mapsto \mathcal{M}(X).$$

Proposition 2.2 is a useful criterion to detect when a ringoid is Morita equivalent to a ring. It is an easy consequence of [13], 8.1.

PROPOSITION 2.2. *If every object in* $\mathbf{R}$ *is a retract of* X *then the evaluation functor* ev_X *is an additive equivalence of abelian categories which restricts to an equivalence between the full subcategories of finitely presented modules.*

In a more categorical language ringoids are defined as categories enriched over the monoidal category of abelian groups with the usual tensor product, compare [4], 6.2. For any ring R one can consider the monoidal category of R-R-bimodules with the R-tensor product and define an *R-ringoid* as a category enriched over it. This is the same as an R-category in the sense of [13] when R is commutative. If $\mathbf{R}$ is an R-ringoid the endomorphism ring $\mathrm{End}_{\mathbf{R}}(X)$ of an object X is in fact an R-algebra. In this case $\mathbf{R}$-modules and morphisms between them take values in the category of (right) R-modules in a natural way.

3. The Algebras $R(E)$ and Related Additive Categories

In this section we give an alternative approach to the algebras $R(E)$ in terms of modules over certain additive categories.

Given a ring R and a set A we write $R\langle A\rangle$ for the free R-module with basis set A. Free R-modules are R-R-bimodules, hence the additive category of free R-modules and right-R-module homomorphisms is an R-ringoid. The *carrier* of an element $x \in R\langle A\rangle$ is the (finite) set $\mathrm{carr}(x) \subset A$ such that $z \in \mathrm{carr}(x)$ if z appears with a non-trivial coefficient in the linear expansion of x.

For every non-empty compact metrizable space E there exists another one X containing E such that the complement $Y = X - E$ is dense in X. The triple $\bar{T} = (X, Y, E)$ can always be chosen to be a tree-like space in the sense of [2], III.1.1. One can also take X to be the (unreduced) cone over E, $X = CE = E \times [0, 1]/E \times \{1\}$. Here we identify E with $E \times \{0\}$ inside the cone CE.

A *free $\bar{T}$-controlled R-module* $R\langle A\rangle_\alpha$ is a free R-module $R\langle A\rangle$ together with a function $\alpha : A \to Y$, called *height function*, such that $\alpha^{-1}(K)$ is finite for every compact subspace $K \subset Y$. The set A is necessarily countable and the derived set of $\alpha(A)$ in X satisfies $\alpha(A)' \subset E$. This derived set is called the *support* of $R\langle A\rangle_\alpha$. *Controlled homomorphisms* $\varphi : R\langle A\rangle_\alpha \to R\langle B\rangle_\beta$ are homomorphisms between the underlying R-modules such that for every $x \in E$ and every neighborhood U of x in X there exists another neighborhood $V \subset U$ of x in X such that if $a \in A$ satisfies $\alpha(a) \in V$ then $\beta(\mathrm{carr}(\varphi(a))) \subset U$. The category $\mathbf{M}_R(\bar{T})$ of free $\bar{T}$-controlled R-modules and controlled homomorphisms is a small additive category. Moreover, it is an R-ringoid. The sum and R-actions on morphism sets are given by those of the underlying free R-module homomorphisms, and the direct sum of two objects is $R\langle A\rangle_\alpha \oplus R\langle B\rangle_\beta = R\langle A \sqcup B\rangle_{(\alpha,\beta)}$, where $A \sqcup B$ is the disjoint union of sets and $(\alpha, \beta) : A \sqcup B \to Y$ is defined as α over A and β over B.

Remark 3.1. The category $\mathbf{M}_R(\bar{T})$ is defined in [2] III.4.7 for $\bar{T}$ a tree-like space. However, as it is pointed out in the Remark after that definition, it is equivalent to the category $\mathcal{B}(X, E; R)$ in [5]. In particular $\mathbf{M}_R(\bar{T})$ only depends on E up to equivalence of categories preserving supports of objects (in fact equivalence of R-ringoids), see 1.23 and 1.24 in [5].

Proposition 3.2 shows that free $\bar{T}$-controlled R-modules are classified by the underlying R-module and the support.

PROPOSITION 3.2. *Two free $\bar{T}$-controlled R-modules $R\langle A\rangle_\alpha$, $R\langle B\rangle_\beta$ are isomorphic if and only if the next two conditions are satisfied:*

(1) *The underlying R-modules are isomorphic $R\langle A\rangle \simeq R\langle B\rangle$,*
(2) *both have the same support $\alpha(A)' = \beta(B)'$.*

If the supports are non-empty then condition (1) *is automatically satisfied. Furthermore, any compact subset $K \subset E$ is the support of some free $\bar{T}$-controlled R-module.*

In the proof of this proposition we shall use the following.

LEMMA 3.3. *Given an injective controlled homomorphism $\varphi : R\langle A\rangle_\alpha \to R\langle B\rangle_\beta$ we have that $\alpha(A)' \subset \beta(B)'$.*

Proof. For any $e \in \alpha(A)'$ we can take a sequence $\{a_n\}_{n\in\mathbb{N}} \subset A$ with $\lim\limits_{n\to\infty} \alpha(a_n) = e$. Since φ is injective $\mathrm{carr}(\varphi(a_n))$ is non-empty for every $n \in \mathbb{N}$ so we can take elements $b_n \in \mathrm{carr}(\varphi(a_n))$. By definition of controlled homomorphism $\lim\limits_{n\to\infty} \beta(b_n) = e$, hence $e \in \beta(B)'$ and the inclusion holds. □

Proof of Proposition 3.2. The case $R = \mathbb{Z}$ and $\bar{T}$ a tree-like space follows from [2], III.4.8 and III.4.16. In general condition (1) is necessary since an isomorphism of free $\bar{T}$-controlled R-modules is also an isomorphism between the underlying R-modules. Moreover, condition (2) is necessary by Lemma 3.3. By Remark 3.1 it is enough to make the proof for tree-like spaces. In the rest of the proof we shall suppose that $\bar{T}$ is tree-like.

If $\alpha(A)' = \alpha(B)' = \emptyset$ then A and B are both finite and any isomorphism $R\langle A\rangle \simeq R\langle B\rangle$ is a controlled isomorphism $R\langle A\rangle_\alpha \simeq R\langle B\rangle_\beta$. If $\alpha(A)' = \beta(B)' \neq \emptyset$ then A and B are infinite countable, so $\mathbb{Z}\langle A\rangle \simeq \mathbb{Z}\langle B\rangle$. Since the proposition holds for $R=\mathbb{Z}$ there is a controlled isomorphism φ: $\mathbb{Z}\langle A\rangle_\alpha \simeq \mathbb{Z}\langle B\rangle_\beta$. Now one can check that $\varphi\otimes R : R\langle A\rangle_\alpha \simeq R\langle B\rangle_\beta$ is an isomorphism of free $\bar{T}$-controlled R-modules.

Finally, given $K\subset E$ compact, if $\mathbb{Z}\langle C\rangle_\gamma$ is a free $\bar{T}$-controlled $\mathbb{Z}$-module whose support is K then the support of $R\langle C\rangle_\gamma$ is K as well, because it only depends on γ. The proof is now complete. □

The next result follows directly from Remark 3.1, Proposition 3.2 and the definition of controlled homomorphisms.

PROPOSITION 3.4. *Up to isomorphism, the endomorphism algebra of a free $\bar{T}$-controlled R-module with support E only depends on E. Moreover, it is isomorphic to $R(E)$.*

The last isomorphism of this proposition is given by the fact that the basis of a free $\bar{T}$-controlled R-module $R\langle A\rangle_\alpha$ with support E must be infinite countable, and hence it can be identified with the non-negative integers $A=\mathbb{N}_0$. Moreover, we can suppose that α is the inclusion of a discrete subspace $\alpha : A\subset Y$, exchanging A by $\alpha(A)$ if necessary. Now we are in the same situation as in the beginning of the introduction. We also derive from Proposition 3.4 that the isomorphism class of the R-algebra $R(E)$ only depends on E, as we claimed in the introduction.

PROPOSITION 3.5. *Every free $\bar{T}$-controlled R-module is a retract of any object whose support is E.*

Proof. Recall from Proposition 3.2 that all objects with support E are isomorphic. By Remark 3.1 it is enough to check the proposition for $\bar{T}$ a tree-like space. For $R=\mathbb{Z}$ and $\bar{T}$ tree-like this proposition is contained in the proof of [2], V.3.4. The result for arbitrary rings follows from the special case $R=\mathbb{Z}$. More precisely, given a free $\bar{T}$-controlled R-module $R\langle A\rangle_\alpha$, let $R\langle B\rangle_\beta$ be chosen with support E. Then $\mathbb{Z}\langle A\rangle_\alpha$ is a retract of $\mathbb{Z}\langle B\rangle_\beta$ (the supports only depend on the height functions) hence we obtain a retraction of $R\langle B\rangle_\beta$ onto $R\langle A\rangle_\alpha$ by tensoring by R. □

As a consequence of this Proposition we get by Proposition 2.2 the following equivalence of categories which will be used from now on as an identification.

COROLLARY 3.6. *The evaluation functor in a free $\bar{T}$-controlled R-module with support E induces an additive equivalence of abelian categories $\mathbf{mod}(\mathbf{M}_R(\bar{T}))\simeq\mathbf{mod}(R(E))$ which restricts to another one $\mathbf{fp}(\mathbf{M}_R(\bar{T}))\simeq\mathbf{fp}(R(E))$.*

Remark 3.7. If $R \simeq R^{op}$, in particular if R is commutative, the transposition of matrices and an explicit isomorphism $R \simeq R^{op}$ induce isomorphisms of R-ringoids $\mathbf{M}_R(\bar{T}) \simeq \mathbf{M}_R(\bar{T})^{op}$ (preserving objects) and R-algebras $R(E) \simeq R(E)^{op}$, compare [5], so in this case right-modules over $\mathbf{M}_R(\bar{T})$ or $R(E)$ are the same as left-modules.

In Proposition 3.8 we compute the dimension of the k-algebra $k(E)$, k any field.

PROPOSITION 3.8. $\dim k(E) = (\text{card } k)^{\aleph_0}$.

Proof. The k-vector space of all $\mathbb{N}_0 \times \mathbb{N}_0$ matrices is the direct product of $\mathbb{N}_0 \times \mathbb{N}_0$ copies of k, and it is known that $\dim \prod_{\mathbb{N}_0 \times \mathbb{N}_0} k = (\text{card } k)^{\aleph_0}$ (see [11], IX.5), hence $\dim k(E) \leqslant (\text{card } k)^{\aleph_0}$. Moreover, for any element $(\mathsf{a}_n)_{n \in \mathbb{N}_0} \in \prod_{\mathbb{N}_0} k$ the diagonal matrix $(\mathsf{b}_{ij})_{i,j} \in \mathbb{N}_0$ with $\mathsf{b}_{nn} = \mathsf{a}_n$belongs to $k(E)$, therefore $k(E)$ has a vector subspace isomorphic to $\prod_{\mathbb{N}_0} k$, and $\dim \prod_{\mathbb{N}_0} k = \text{card } k^{\aleph_0}$ as well, so the equality of the statement holds. □

3.1. THE SPECIAL CASE card E FINITE

If card E is finite, since E is metrizable, it must have the discrete topology, so E is the discrete set $\boldsymbol{n}$ with $n \times$ card E points. For this particular space we can take a tree-like space $\bar{T}_n = (\hat{T}_n, T_n, \boldsymbol{n})$, where T_n is a locally compact tree with n Freudenthal ends and $\hat{T}_n$ is the Freudenthal compactification of T_n (see [2], III.1.3). Moreover, if T_n^0 is the vertex set of T_n and $\delta : T_n^0 \subset T_n$ is the inclusion, the support of $R\langle T_n^0 \rangle_\delta$ is $\boldsymbol{n}$, in particular $R(\boldsymbol{n})$ is the endomorphism ring of this object. Let us fix the following particular tree T_n: the vertex set of T_n is

$$T_n^0 = \{v_0\} \cup \{v_m^1, \ldots, v_m^n\}_{m \geqslant 1},$$

and there are edges joining v_0 with v_1^i and v_m^i with v_{m+1}^i ($1 \leqslant i \leqslant n$, $m \geqslant 1$). The additive category $\mathbf{M}_R(\bar{T}_n)$ is equivalent to the full subcategory of objects $R\langle A \rangle_\alpha$ such that $\alpha(A) \subset T_n^0$, compare the proof of [2], V.3.4. From now on we shall always work in this subcategory, and we denote it by $\mathbf{M}_R(\bar{T}_n)$ as well.

We are going to give an alternative description for controlled homomorphisms in $\mathbf{M}_R(\bar{T}_n)$. For this we define the following sets for any height function $\alpha : A \to T_n^0 \subset T_n$ ($1 \leqslant i \leqslant n$, $j \geqslant 1$)

$$A_j^i = \bigcup_{l \geqslant j} \alpha^{-1}(v_l^i).$$

A morphism $\varphi : R\langle B \rangle_\beta \to R\langle A \rangle_\alpha$ in $\mathbf{M}_R(\bar{T}_n)$ is controlled if and only if for every $m \geqslant 1$ there exists $M \geqslant 1$ such that $\varphi(B_M^i) \subset R\langle A_m^i \rangle$ for any $1 \leqslant i \leqslant n$.

We shall omit the superindex i when $n=1$. Moreover, for the next sections we fix the following notation ($m \geqslant 0$)

$$ {}_mA = \alpha^{-1}(v_0) \cup \left[\bigcup_{1 \leqslant i \leqslant n} \bigcup_{l \leqslant m} \alpha^{-1}(v_l^i) \right], \qquad {}_mA_j^i = {}_mA \cap A_j^i. $$

Remark 3.9. If $n=1, T_1=[0,+\infty)$ is the half-line and $T_1^0=\mathbb{N}_0$ the non-negative integers. Moreover $R(\boldsymbol{I})$ is the R-algebra RCFM(R) of $\mathbb{N}_0 \times \mathbb{N}_0$ matrices with entries in R such that every row and every column has a finite number of non-zero entries (*row×column×finite matrices*), compare [2], V.3.8.

Remark 3.10. If E is any compact metrizable space with at least n points we can fully include $\mathbf{M}_R(\bar{T}_n)$ into $\mathbf{M}_R(\bar{T})$. For this we only need to take n disjoint sequences $\{v_m^i\}_{m \geqslant 1}$ ($1 \leqslant i \leqslant n$) contained in Y converging in X to n different points belonging to E, and an additional point $v_0 \in Y$ out of the sequences. Now we identify $\mathbf{M}_R(\bar{T}_n)$ with the full subcategory of $\mathbf{M}_R(\bar{T})$ given by objects $R\langle A\rangle_\alpha$ with $\alpha(A) \subset \{v_0\} \cup \{v_m^1, \ldots, v_m^n\}_{m \geqslant 1}$. If we write $\mathbb{F}$ for this full inclusion, we get another full inclusion $\mathbb{F}_*: \mathbf{mod}(\mathbf{M}_R(\bar{T}_n)) \to \mathbf{mod}(\mathbf{M}_R(\bar{T}))$ together with a retraction up to natural equivalence $\mathbb{F}^*: \mathbf{mod}(\mathbf{M}_R(\bar{T})) \to \mathbf{mod}(\mathbf{M}_R(\bar{T}_n))$. Moreover, the first functor $\mathbb{F}_*$ restricts to the full subcategories of f. p. modules, see Section 2, hence the decomposition problem for f. p. $R(\boldsymbol{n})$-modules is included in the decomposition problem for f. p. $R(E)$-modules, in particular we only need to prove Theorem 1.1 for card E finite.

4. Countably Presented *R*-Modules as Finitely Presented RCFM(*R*)-Modules

There is a full exact inclusion of abelian categories

$$ \mathfrak{i}: \mathbf{mod}(R) \to \mathbf{mod}(\mathrm{RCFM}(R)) $$

defined by $\mathfrak{i}M = \mathrm{Hom}_R(R\langle\mathbb{N}_0\rangle, M)$. The ring RCFM($R$) acts on $\mathfrak{i}M$ by endomorphisms of $R\langle\mathbb{N}_0\rangle$.

Let $\mathfrak{f}: \mathbf{M}_R(\bar{T}_1) \to \mathbf{mod}(R)$ be the forgetful functor which sends a free $\bar{T}_1$-controlled R-module to its underlying R-module. The RCFM(R)-module $\mathfrak{i}M$ can be regarded as the functor $\mathfrak{i}M = \mathrm{Hom}_R(\mathfrak{f}, M): \mathbf{M}_R(\bar{T}_1)^{op} \to \mathbf{Ab}$.

PROPOSITION 4.1. *The functor* $\mathfrak{i}$ *has an exact left-adjoint* $\mathfrak{r}$ *such that* $\mathfrak{r}\mathfrak{i}$ *is naturally equivalent to the identity functor. Moreover,* $\mathfrak{r}$ *can be chosen to be the evaluation functor in a free* $\bar{T}_1$*-controlled R-module with one generator.*

Proof. Let $R\langle e\rangle_\phi$ be a free $\bar{T}_1$-controlled R-module whose basis is a singleton $\{e\}$ (all these objects are isomorphic in $\mathbf{M}_R(\bar{T}_1)$ by Propostion 3.2). The endomorphism ring of this object is R, hence the evaluation functor $ev_{R\langle e\rangle_\phi}$ takes values in the category of R-modules. Let us see that the exact functor $ev_{R\langle e\rangle_\phi}$ is left-adjoint to $\mathfrak{i}$. A left-adjoint for $\mathfrak{i}$ exists as a consequence of the adjoint functor theorem, see [3], 3.3.7, so we have just to check that $\mathrm{Hom}_{\mathbf{M}_R(\bar{T}_1)}(R\langle A\rangle_\alpha, \mathfrak{i}M)=\mathrm{Hom}_R(ev_{R\langle e\rangle_\phi}R\langle A\rangle_\alpha, M)$ for any free $\bar{T}_1$-controlled R-module $R\langle A\rangle_\alpha$ in a natural way. This follows from the obvious natural identification $ev_{R\langle e\rangle_\phi}R\langle A\rangle_\alpha=R\langle A\rangle$ and Yoneda's lemma. □

COROLLARY 4.2. *If M is an R-module and $\mathcal{N}$ an* RCFM(R)*-module then there are natural isomorphisms* $(n\geqslant 0)$

$$\mathrm{Ext}^n_{R(E)}(\mathcal{N}, \mathfrak{i}M)\simeq \mathrm{Ext}^n_R(\mathfrak{r}\mathcal{N}, M).$$

In particular if $R=k$ is a field the RCFM(k)*-modules $\mathfrak{i}M$ are all injective.*

An R-module is *countably presented* provided it is the cokernel of a morphism between free R-modules with countable basis. Obviously the cokernel of a morphism between countably presented R-modules is countably presented as well. In particular direct summands of countably presented R-modules are countably presented.

PROPOSITION 4.3. *The functor $\mathfrak{i}$ sends countably presented R-modules to finitely presented* RCFM(R)*-modules.*

In the proof of this proposition we shall use the row-column-finite matrices A and B defined by

- $\mathsf{a}_{i+1,i}=1$ $(i\in\mathbb{N}_0)$ and $\mathsf{a}_{ij}=0$ in other cases,
- $\mathsf{b}_{\frac{n(n+1)}{2}+i,\frac{(n-1)n}{2}+i}=1$ for any $n>0$ and $0\leqslant i<n$, and $\mathsf{b}_{ij}=0$ otherwise.

And we regard RCFM(R) as the endomorphism R-algebra of the free $\bar{T}_1$-controlled R-module $R\langle\mathbb{N}_0\rangle_\delta$, where $\delta:\mathbb{N}_0\subset[0,+\infty)$ is the inclusion, see Remark 3.9 and Proposition 3.4.

Proof of Proposition 4.3. Since $\mathfrak{i}$ is exact it will be enough to check the proposition for the countably presented R-modules R and $R\langle\mathbb{N}_0\rangle$. Recall that $\mathrm{Hom}_{\mathbf{M}_R(\bar{T}_1)}(R\langle\mathbb{N}_0\rangle_\delta, \mathfrak{i}M)=\mathrm{Hom}_R(R\langle\mathbb{N}_0\rangle, M)$ for any R-module M. The RCFM(R)-modules $\mathfrak{i}R$ and $\mathfrak{i}R\langle\mathbb{N}_0\rangle$ are the cokernels of $(\mathsf{I}-\mathsf{A})$ and $(\mathsf{I}-\mathsf{B})$, respectively. The natural projections onto the cokernel are given by the homomorphisms $R\langle\mathbb{N}_0\rangle\to R$ and $R\langle\mathbb{N}_0\rangle\to R\langle\mathbb{N}_0\rangle$ defined on generators by $n\mapsto 1$ $(n\geqslant 0)$ and $(n(n+1)/2)+i\mapsto i$ $(n\geqslant i\geqslant 0)$, respectively. □

As a consequence of Propositions 4.1 and 4.3, and Remark 3.10 we get Corollary 4.4.

COROLLARY 4.4. *The representation problem for countably generated R-modules is embedded in the representation problem for finitely presented $R(E)$-modules, where E is any non-empty compact metrizable space.*

5. A Review on Pro-categories

Here we recall the definition of pro-objects and pro-morphisms and some of its basic facts. We refer to [6, 12] for further results on this subject used in this paper.

Any partially ordered set (poset) Λ can be regarded as a small category with a unique morphism $u \to v$ provided $u \geqslant v, u, v \in \Lambda$. A poset Λ is *directed* if given $u, v \in \Lambda$ there exists $w \in \Lambda$ with $w \geqslant u, v$. Moreover, Λ is *cofinite* if the set $\{u \in \Lambda; u \leqslant v\}$ is finite for every $v \in \Lambda$.

A *pro-object* or *inverse system* $X_\bullet$ over a category $\mathbf{C}$ is a functor $X_\bullet : \Lambda \to \mathbf{C}$, where Λ is a directed cofinite poset. If $u \in \Lambda$ we usually write $X_u = X_\bullet(u)$. The morphisms $X_\bullet(u \to v)$ $(u, v \in \Lambda, u \geqslant v)$ are the *bonding morphisms* of $X_\bullet$, and Λ is the *indexing set* of the inverse system.

The category pro-$\mathbf{C}$ has objects inverse systems over $\mathbf{C}$. Morphism sets are given by the following formula

$$\mathrm{Hom}_{\text{pro-}\mathbf{C}}(X_\bullet, Y_\bullet) = \lim_{v} \operatorname*{colim}_{u} \mathrm{Hom}_{\mathbf{C}}(X_u, Y_v). \tag{5.a}$$

We identify any object in $\mathbf{C}$ with the inverse system whose indexing set is a singleton $\Lambda = *$. This defines a full inclusion of categories $\mathbf{C} \subset \text{pro-}\mathbf{C}$. This inclusion has a right-adjoint, the (inverse) limit functor $\lim : \text{pro-}\mathbf{C} \to \mathbf{C}$, $\lim X_\bullet = \lim_u X_u$.

The category pro-$\mathbf{C}$ is abelian whenever $\mathbf{C}$ is, see [6], 6.4. This will be always the case, because we are only going to use in this context the category $\mathbf{C} = \mathbf{mod}(k)$ of k-vector spaces. If V is a vector space and $X_\bullet$ an inverse system of vector spaces, then by (5.a)

$$\mathrm{Hom}_{\text{pro-}\mathbf{mod}(k)}(V, X_\bullet) = \lim_{v} \mathrm{Hom}_k(V, X_v) = \mathrm{Hom}_k(V, \lim X_\bullet).$$

Hence, since $\mathrm{Hom}_k(V, -)$ is an exact functor in the category of vector spaces, the Grothendieck spectral sequence (see [10], 9.3) yields an isomorphism

$$\mathrm{Ext}^1_{\text{pro-}\mathbf{mod}(k)}(V, X_\bullet) = \mathrm{Hom}_k(V, \lim{}^1 X_\bullet). \tag{5.b}$$

6. Numerical Invariants of Finitely Presented $k(\boldsymbol{n})$-Modules

In this section we shall define invariants of the isomorphism class of a f. p. $k(\boldsymbol{n})$- module lying in the abelian monoids $\mathbb{N}_{\infty,n}$ $(n \geqslant 1)$. The abelian monoid $\mathbb{N}_{\infty,n}$ has $n+1$ generators

$$1, \infty_1, \dots, \infty_n$$

and $2n$ relations

$$1+\infty_i=\infty_i, \qquad \infty_i+\infty_i=\infty_i, \quad (1 \leqslant i \leqslant n).$$

As a set $\mathbb{N}_{\infty,n}$ is

$$\mathbb{N}_{\infty,n}=\mathbb{N}_0 \sqcup \left\{\infty_S=\sum_{i\in S}\infty_i;\, \emptyset \neq S \subset \{1,\dots n\}\right\}.$$

For the sake of simplicity if $n=1$ we write $\mathbb{N}_{\infty,1}=\mathbb{N}_\infty$ and $\infty_1=\infty$. The notation of Section 3.1 will be used without any further mention.

Let $\varphi: k\langle B\rangle_\beta \to k(A)_\alpha$ be a morphism in $\mathbf{M}_k(\bar{T}_n)$. We define the element $\lambda_\varphi \in \mathbb{N}_{\infty,n}$ in the following way: if the vector space

$$L_\varphi=\frac{k\langle A\rangle}{\bigcap_{m\geqslant 1}\sum_{i=1}^n[k\langle A_m^i\rangle+\varphi(k\langle B\rangle)]}$$

is finite-dimensional then $\lambda_\varphi=\dim L_\varphi$, otherwise $\lambda_\varphi=\infty_S$, where $S \subset \{1,\dots,n\}$ is the biggest subset such that if $i \notin S$ then there exists $M \geqslant 1$ with $k\langle A_M^i\rangle \subset \varphi(k\langle B\rangle)$.

PROPOSITION 6.1. *The element λ_φ only depends on the isomorphism class of the f.p. $k(\boldsymbol{n})$-module* Coker φ, *and $\lambda_{\varphi\oplus\psi}=\lambda_\varphi+\lambda_\psi$.*

Proof. One can readily check by using the alternative description of controlled homomorphisms given in Section 3.1 that the correspondence $\varphi \mapsto \Upsilon(\varphi)=L_\varphi$ defines an additive functor Υ from $\mathbf{pair}(\mathbf{M}_k(\bar{T}_n))$ to the category of k-vector spaces. Moreover, if $V_\bullet^{\varphi,i}$ is the inverse system of k-vector spaces indexed by $\mathbb{N}$ given by $(1 \leqslant i \leqslant n)$

$$V_m^{\varphi,i}=\frac{k\langle A_m^i\rangle+\varphi(k\langle B\rangle)}{\varphi(k\langle B\rangle)},$$

and the obvious inclusions as bonding morphisms, the correspondences $\varphi \mapsto \Theta_i(\varphi)=V_\bullet^{\varphi,i}$ also define additive functors Θ_i from $\mathbf{pair}(\mathbf{M}_k(\bar{T}_n))$ to the category of pro-vector spaces. Furthermore, it is easy to see that the functors Υ, Θ_i $(1 \leqslant i \leqslant n)$ factor through the natural equivalence relation $\sim$ in $\mathbf{pair}(\mathbf{M}_k(\bar{T}_n))$, hence the first statement of the proposition follows from

Proposition 2.1 and the fact that λ_φ is defined as $\dim\Upsilon(\varphi)$ provided this vector space is finite-dimensional, and otherwise $\lambda_\varphi = \infty_S$, where $S = \{i \in \{1, \dots, n\}; \Theta_i(\varphi) \not\cong 0\}$. The second part of the statement follows from the additivity of the functors Υ, Θ_i $(1 \leqslant i \leqslant n)$. □

If the following vector space has finite dimension,

$$M_\varphi^i = \frac{\bigcap_{m\geqslant 1}\left[k\langle A_m^i\rangle + \varphi(k\langle B\rangle)\right]}{\bigcap_{m\geqslant 1}\left\{\left[k\langle A_m^i\rangle + \varphi(k\langle B\rangle)\right] \cap \left[\sum_{j\neq i} k\langle A_m^j\rangle + \varphi(k\langle B\rangle)\right]\right\}},$$

the element $\mu_\varphi^i \in \mathbb{N}_\infty$ $(1 \leqslant i \leqslant n)$ is defined as $\mu_\varphi^i = \dim M_\varphi^i$, otherwise $\mu_\varphi^i = \infty$.

PROPOSITION 6.2. *The elements* μ_φ^i $(1 \leqslant i \leqslant n)$ *only depend on the isomorphism class of the f.p.* $k(\boldsymbol{n})$*-module* Coker φ, *and* $\mu_{\varphi\oplus\psi}^i = \mu_\varphi^i + \mu_\psi^i$.

Proof. By using the characterization of controlled homomorphisms given in Section 3.1 one readily checks that the correspondences $\varphi \mapsto M_\varphi^i$ define additive functors from $\mathbf{pair}(\mathbf{M}_k(\bar{T}_n))$ to the category of k-vector spaces. Moreover, these functors factor through the natural equivalence relation $\sim$, hence the proposition follows from Proposition 2.1. □

In order to define elements $\nu_\varphi^i \in \mathbb{N}_\infty$ $(1 \leqslant i \leqslant n)$ we introduce inverse systems of k-vector spaces $U_\bullet^{\varphi,i}$, indexed by the set $\mathbb{N}\times\mathbb{N}$ with the product partial order, given by

$$U_{pq}^{\varphi,i} = \frac{k\langle A_p^i\rangle \cap \varphi(k\langle B\rangle)}{k\langle A_p^i\rangle \cap \varphi(k\langle B_q^i\rangle)},$$

and bonding homomorphisms induced by the obvious inclusions of vector spaces. If the limit of $U_\bullet^{\varphi,i}$ is finite-dimensional we set $\nu_\varphi^i = \dim \lim U_\bullet^{\varphi,i}$, otherwise $\nu_\varphi^i = \infty$.

PROPOSITION 6.3. *The elements* ν_φ^i $(1 \leqslant i \leqslant n)$ *only depend on the isomorphism class of the f.p.* $k(\boldsymbol{n})$*-module* Coker φ, *and* $\nu_{\varphi\oplus\psi}^i = \nu_\varphi^i + \nu_\psi^i$.

Proof. One can check by using the description of controlled homomorphisms given in Section 3.1 that the correspondences $\varphi \mapsto U_\bullet^{\varphi,i}$ are additive functors from $\mathbf{pair}(\mathbf{M}_k(\bar{T}_n))$ to the category of pro-vector spaces, and these functors factor through the natural equivalence relation $\sim$, hence the proposition follows from Proposition 2.1. □

Propositions 6.1–6.3 are summarized in the following corollary.

COROLLARY 6.4. *There are well defined morphisms of abelian monoids* ($n \in \mathbb{N}$)

$$\Phi_n : \mathrm{Iso}(\mathbf{fp}(k(\boldsymbol{n}))) \to \mathbb{N}_{\infty,n} \times \prod_{i=1}^{n} \mathbb{N}_\infty \times \prod_{i=1}^{n} \mathbb{N}_\infty$$

which send the isomorphism class $[\mathcal{M}]$ *of a f. p.* $k(\boldsymbol{n})$*-module* $\mathcal{M} =$ Coker φ *to*

$$\Phi_n([\mathcal{M}]) = \left(\lambda_\varphi, \left(\mu_\varphi^i\right)_{i=1}^n, \left(\nu_\varphi^i\right)_{i=1}^n\right).$$

From now on we shall write $\lambda_{\mathcal{M}} = \lambda_\varphi$, $\mu^i_{\mathcal{M}} = \mu^i_\varphi$ and $\nu^i_{\mathcal{M}} = \nu^i_\varphi$ $(1 \leqslant i \leqslant n)$ if $\mathcal{M} \times$ Coker φ and omit the superscript i when $n = 1$.

Remark 6.5. There are n full inclusions $\mathbb{F}^i : \mathbf{M}_k(\bar{T}_1) \to \mathbf{M}_k(\bar{T}_n)$ $(1 \leqslant i \leqslant n)$ defined by identifying $T_1^0 = \mathbb{N}_0$ (see 3.9) with the subset $\{v_0\} \cup \{v^i_m\}_{m \geqslant 1} \subset T_n^0$ in the obvious way, (see Remark 3.10).

Proposition 6.6 can be easily checked by using the commutativity of (2.a) and the right-exactness of the functors $\mathbb{F}^i_*$.

PROPOSITION 6.6. *If* $\mathcal{M}$ *is a f. p.* $k(\mathit{1})$*-module then for every* $1 \leqslant i \leqslant n$

- $\lambda_{\mathbb{F}^i_*\mathcal{M}} = \lambda_{\mathcal{M}}$ *if* $\lambda_{\mathcal{M}} \in \mathbb{N}_0$*, and* $\lambda_{\mathbb{F}^i_*\mathcal{M}} = \infty_i$ *otherwise,*
- $\mu^i_{\mathbb{F}^i_*\mathcal{M}} = \mu_{\mathcal{M}}$ *and* $\mu^j_{\mathbb{F}^i_*\mathcal{M}} = 0$ *if* $j \neq i$*,*
- $\nu^i_{\mathbb{F}^i_*\mathcal{M}} = \nu_{\mathcal{M}}$ *and* $\nu^j_{\mathbb{F}^i_*\mathcal{M}} = 0$ *if* $j \neq i$*.*

7. Classification of Finitely Presented RCFM(k)-Modules

Recall from Remark 3.9 that the k-algebra $k(\mathit{1})$ coincides with RCFM(k), the k-algebra of matrices $\mathsf{A} = (\mathsf{a}_{ij})_{i,j\in\mathbb{N}_0}$ with entries in k such that every row and every column has at most a finite number of non-trivial entries (row-column-finite matrices). Those matrices are the endomorphisms of the free $\bar{T}_1$-controlled k-vector space $k\langle\mathbb{N}_0\rangle_\delta$, where $\delta : \mathbb{N}_0 \subset [0, +\infty)$ is the inclusion of the vertex set. The unit element of the k-algebra RCFM(k) is the identity matrix I with $\mathsf{i}_{ii} = 1$ $(i \in \mathbb{N}_0)$ and $\mathsf{i}_{ij} = 0$ if $i \neq j$. For the sake of simplicity we abbreviate $\mathcal{R} =$RCFM(k).

Consider the matrices A and B used in the proof of Proposition 4.3, they are defined as

- $\mathsf{a}_{i+1,i} = 1$ $(i \in \mathbb{N}_0)$ and $\mathsf{a}_{ij} = 0$ in other cases,
- $\mathsf{b}_{\frac{n(n+1)}{2}+i,\frac{(n-1)n}{2}+i} = 1$ for any $n > 0$ and $0 \leqslant i < n$, and $\mathsf{b}_{ij} = 0$ otherwise.

We define the following $\mathcal{R}$-modules

- $\mathcal{A} = \mathcal{R}/(\mathsf{A}\mathcal{R})$,
- $\mathcal{B} = \mathcal{R}/((\mathsf{I} - \mathsf{A})\mathcal{R})$,
- $\mathcal{C} = \mathcal{R}/((\mathsf{I} - \mathsf{A}^t)\mathcal{R})$,
- $\mathcal{B}_\infty = \mathcal{R}/((\mathsf{I} - \mathsf{B})\mathcal{R})$,
- $\mathcal{C}_\infty = \mathcal{R}/((\mathsf{I} - \mathsf{B}^t)\mathcal{R})$.

The main result of this section is as follows.

THEOREM 7.1. (Classification of f. p. RCFM(k)-modules). *There is a solution to the decomposition problem in the category of f. p.* RCFM(k)-*modules given by the following elementary modules*

$$\mathcal{A}, \mathcal{R}, \mathcal{B}, \mathcal{B}_\infty, \mathcal{C}, \mathcal{C}_\infty,$$

and elementary isomorphisms

$$\mathcal{A} \oplus \mathcal{R} \simeq \mathcal{R}, \qquad \mathcal{R} \oplus \mathcal{R} \simeq \mathcal{R}, \qquad \mathcal{B} \oplus \mathcal{B}_\infty \simeq \mathcal{B}_\infty,$$
$$\mathcal{B}_\infty \oplus \mathcal{B}_\infty \simeq \mathcal{B}_\infty, \qquad \mathcal{C} \oplus \mathcal{C}_\infty \simeq \mathcal{C}_\infty, \qquad \mathcal{C}_\infty \oplus \mathcal{C}_\infty \simeq \mathcal{C}_\infty.$$

This theorem implies Theorems 1.1 and 1.2 for card $E = 1$. It is a direct consequence of the next two results. We shall use the following notation. Given $n \in \mathbb{N}_0$ we write $\mathcal{A}_n, \mathcal{B}_n$ and $\mathcal{C}_n$ for the direct sum of n copies of $\mathcal{A}, \mathcal{B}$ or $\mathcal{C}$, respectively, and $\mathcal{A}_\infty = \mathcal{R}$.

THEOREM 7.2. *For every f. p. $\mathcal{R}$-module $\mathcal{M}$ there is an isomorphism*

$$\mathcal{M} \simeq \mathcal{A}_{\lambda_{\mathcal{M}}} \oplus \mathcal{B}_{\mu_{\mathcal{M}}} \oplus \mathcal{C}_{\nu_{\mathcal{M}}}.$$

PROPOSITION 7.3. *We have the following equalities*:

- $\Phi_1(\mathcal{A}) = (1, 0, 0)$,
- $\Phi_1(\mathcal{B}) = (0, 1, 0)$,
- $\Phi_1(\mathcal{C}) = (0, 0, 1)$,
- $\Phi_1(\mathcal{R}) = (\infty, 0, 0)$,
- $\Phi_1(\mathcal{B}_\infty) = (0, \infty, 0)$,
- $\Phi_1(\mathcal{C}_\infty) = (0, 0, \infty)$.

In particular we have that the following corollary.

COROLLARY 7.4. *The monoid morphism*

$$\Phi_1 : \mathrm{Iso}(\mathbf{fp}(k(\mathit{1}))) \to \mathbb{N}_\infty \times \mathbb{N}_\infty \times \mathbb{N}_\infty$$

is an isomorphism.

The proof of Proposition 7.3 will be given later. Theorem 7.2 is a direct consequence of Lemmas 7.18–7.20 and 7.22. They are the hardest technical

results of this paper. In fact the rest of this long section is highly technical. It is focused towards proving Theorem 7.2, although some interesting corollaries on the homological algebra of finitely presented $\mathcal{R}$-modules are derived from the technical lemmas. These homological results are used in the proof of Theorem 7.2 as well as in the appendix. The reader can skip this material in a first reading. We shall use the notation from Section 3.1 with no explicit mention.
The next result is an easy computation.

LEMMA 7.5. *The following equalities hold in* $\mathcal{R}$:

(1) $\mathsf{A}^t\mathsf{A} = \mathsf{I}$,
(2) $\mathsf{B}^t\mathsf{B} = \mathsf{I}$.

Lemma 7.6 follows directly from Proposition 3.2.

LEMMA 7.6. *Two free* $\bar{T}_1$*-controlled* k*-vector spaces* $k\langle A\rangle_\alpha, k\langle B\rangle_\beta$ *are isomorphic in* $\mathbf{M}_k(\bar{T}_1)$ *if and only if* A *and* B *have the same cardinality.*

LEMMA 7.7. *The* $\mathcal{R}$*-module* $\mathcal{A}$ *is isomorphic to any* 1*-dimensional free* $\bar{T}_1$*-controlled* k*-vector space.*

Proof. If $k\langle e\rangle_\phi$ is a 1-dimensional free $\bar{T}_1$-controlled k-vector space, the cokernel of A is given by the controlled homomorphism $\varphi : k\langle\mathbb{N}_0\rangle_\delta \twoheadrightarrow k\langle e\rangle_\phi$ defined over the basic elements as $0 \mapsto e$ and $n \mapsto 0$ for $n > 0$. □

The proof of Proposition 7.3 is as follows.

Proof of Proposition 7.3. In this proof we omit some straightforward but tedious computations which can be carried out by the interested reader with not too much dificulty. We shall write $\mathbb{N}_p$ $(p \geqslant 1)$ for the set of naturals $\geqslant p$.

The $\mathcal{R}$-module $\mathcal{R}$ corresponds to the free $\bar{T}_1$-controlled k-vector space $k\langle\mathbb{N}_0\rangle_\delta$, where $\delta : \mathbb{N}_0 \subset [0, +\infty)$ is the inclusion, hence it is the cokernel of the trivial morphism $0 : 0 \to k\langle\mathbb{N}_0\rangle_\delta$, and the equalities $\lambda_{\mathcal{R}} = \infty, \mu_{\mathcal{R}} = 0$ hold immediately, moreover, $U^0_{m,n} = k\langle\mathbb{N}_m\rangle$ for all $m, n \in \mathbb{N}$, and given $M \geqslant m, N \geqslant n$ the corresponding bonding homomorphism in $U^0_\bullet$ is the inclusion $U^0_{M,N} = k\langle\mathbb{N}_M\rangle \subset k\langle\mathbb{N}_m\rangle = U^0_{m,n}$, therefore $\lim U^0_\bullet = \bigcap_{m\in\mathbb{N}} k\langle\mathbb{N}_m\rangle = 0$ and $\nu_{\mathcal{R}} = 0$.

By Lemma 7.7 the $\mathbf{M}_k(\bar{T}_1)$-module $\mathcal{A}$ is the cokernel of the trivial morphism $0 \to k\langle e\rangle_\phi$, where $k\langle e\rangle_\phi$ is a 1-dimensional free $\bar{T}_1$-controlled k-vector space, hence the equality $\Phi_1(\mathcal{A}) = (1, 0, 0)$ follows easily.

One can check that $k\langle\mathbb{N}_n\rangle + (\mathsf{I}-\mathsf{A})(k\langle\mathbb{N}_0\rangle) = k\langle\mathbb{N}_0\rangle$ for all $n\in\mathbb{N}$, and $k\langle\mathbb{N}_0\rangle/(\mathsf{I}-\mathsf{A})(k\langle\mathbb{N}_0\rangle)\simeq k$ generated by the class of any $n\in\mathbb{N}_0$, so $\lambda_{\mathcal{B}}=0$ and $\mu_{\mathcal{B}}=1$. Moreover, $k\langle\mathbb{N}_n\rangle\cap(\mathsf{I}-\mathsf{A})(k\langle\mathbb{N}_0\rangle)=(\mathsf{I}-\mathsf{A})(k\langle\mathbb{N}_n\rangle)$ and hence $U^{(\mathsf{I}-\mathsf{A})}_{n,n}=0$ for all $n\in\mathbb{N}$, therefore $\lim U^{(\mathsf{I}-\mathsf{A})}_{\bullet}=0$ since the diagonal subset $\{(n,n);n\in\mathbb{N}\}\subset\mathbb{N}\times\mathbb{N}$ is cofinal, so $\nu_{\mathcal{B}}=0$.

One can see that $(\mathsf{I}-\mathsf{A}^t)(k\langle\mathbb{N}_0\rangle)=k\langle\mathbb{N}_0\rangle$, hence $\lambda_{\mathcal{C}}=0=\mu_{\mathcal{C}}$, moreover $(\mathsf{I}-\mathsf{A}^t)(k\langle\mathbb{N}_n\rangle)$ is generated by the set $\{m-(m-1)\}_{m\geqslant n}$, therefore $U^{(\mathsf{I}-\mathsf{A}^t)}_{n-1,n}\simeq k$ generated by the class of any $m\geqslant n-1$ and the bonding homomorphism $U^{(\mathsf{I}-\mathsf{A}^t)}_{n,n+1}\to U^{(\mathsf{I}-\mathsf{A}^t)}_{n-1,n}$ is an isomorphism, so again by cofinality we see that $\lim U^{(\mathsf{I}-\mathsf{A}^t)}_{\bullet}\simeq k$, in particular $\nu_{\mathcal{C}}=1$.

The vector space $k\langle\mathbb{N}_n\rangle+(\mathsf{I}-\mathsf{B})(k\langle\mathbb{N}_0\rangle)$ is the whole $k\langle\mathbb{N}_0\rangle$, so $\lambda_{\mathcal{B}_\infty}=0$, moreover, a basis of $k\langle\mathbb{N}_0\rangle/(\mathsf{I}-\mathsf{B})(k\langle\mathbb{N}_0\rangle)$ is $\{n(n+3)/2\}_{n\in\mathbb{N}_0}$, hence $\mu_{\mathcal{B}_\infty}=\infty$. One can check that $k\langle\mathbb{N}_n\rangle\cap(\mathsf{I}-\mathsf{B})(k\langle\mathbb{N}_0\rangle)=(\mathsf{I}-\mathsf{B})(k\langle\mathbb{N}_n\rangle)$, therefore $U^{(\mathsf{I}-\mathsf{B})}_{n,n}=0, \lim U^{(\mathsf{I}-\mathsf{B})}_{\bullet}=0$ and $\nu_{\mathcal{B}_\infty}=0$.

Finally $(\mathsf{I}-\mathsf{B}^t)(k\langle\mathbb{N}_0\rangle)=k\langle\mathbb{N}_0\rangle$, so $\lambda_{\mathcal{C}_\infty}=0=\mu_{\mathcal{C}_\infty}$, and there are isomorphisms $(n\in\mathbb{N}_0)$

$$U^{(\mathsf{I}-\mathsf{B}^t)}_{\frac{(n+1)(n+2)}{2},\frac{n(n+1)}{2}}\simeq k\left\langle\frac{n(n+1)}{2},\dots,\frac{n(n+1)}{2}+n\right\rangle,$$

moreover, the following bonding homomorphism $(n>0)$

$$U^{(\mathsf{I}-\mathsf{B}^t)}_{\frac{(n+1)(n+2)}{2},\frac{n(n+1)}{2}}\to U^{(\mathsf{I}-\mathsf{B}^t)}_{\frac{n(n+1)}{2},\frac{(n-1)n}{2}}$$

sends $(n(n+1)/2)+m$ to $((n-1)n/2)+m$ if $m<n$ and $(n(n+1)/2)+n$ to the trivial element, so $\lim U^{(\mathsf{I}-\mathsf{B}^t)}_{\bullet}=\prod_{\mathbb{N}_0}k$ is the direct product of an infinite countable number of copies of k and the equality $\nu_{\mathcal{C}_\infty}=\infty$ holds. □

LEMMA 7.8. *There is an $\mathcal{R}$-module isomorphism $\mathcal{R}/\mathsf{B}\mathcal{R}\simeq\mathcal{R}$.*

Proof. Let $A\subset\mathbb{N}_0$ be the infinite subset $A=\{n(n+3)/2\}_{n\in\mathbb{N}_0}$, and $\alpha: A\subset\mathbb{N}_0$ the inclusion. The next sequence, where φ is the obvious projection, is exact

$$k\langle\mathbb{N}_0\rangle_\delta\overset{B}{\hookrightarrow}k\langle\mathbb{N}_0\rangle_\delta\overset{\varphi}{\twoheadrightarrow}k\langle A\rangle_\alpha.$$

Hence the lemma follows from Lemma 7.6. □

LEMMA 7.9. *Left-multiplication by one of the following matrices induces an injective right-$\mathcal{R}$-module homomorphism $\mathcal{R}\to\mathcal{R}$,*

$$\mathsf{A},(\mathsf{I}-\mathsf{A}),(\mathsf{I}-\mathsf{A}^t),(\mathsf{I}-\mathsf{B}),(\mathsf{I}-\mathsf{B}^t).$$

Proof. The matrix A has a left-inverse in $\mathcal{R}$ by Lemma 7.5. The other matrices have a left-inverse either in the k-algebra CFM(k) of column-finite matrices or in the k-algebra RFM(k) of row-finite matrices. Both k-algebras contain $\mathcal{R}$, moreover, $\mathcal{R}$ = CFM(k)$\cap$RFM(k). The k-algebra CFM(k) is just the endomorphism ring of the k-vector space $k\langle\mathbb{N}_0\rangle$, and there is an isomorphism RFM(k)$\simeq$ CFM(k)^{op} given by transposition. More precisely, let C, D be the matrices in RFM(k) defined by $\mathsf{c}^1_{ij} = 1$ if $i \geqslant j$ and zero otherwise, and $\mathsf{d}_{\frac{(m-1)m}{2}+i, \frac{(n-1)n}{2}+i} = 1$ $(m \geqslant n > i \geqslant 0)$ and trivial in other cases. One can check that $\mathsf{C}(\mathsf{I}-\mathsf{A}) = \mathsf{I}, \mathsf{C}^t(\mathsf{I}-\mathsf{A}^t) = \mathsf{I}, \mathsf{D}(\mathsf{I}-\mathsf{B}) = \mathsf{I}$ and $\mathsf{D}(\mathsf{I}-\mathsf{B}^t) = \mathsf{I}$, hence the lemma follows. □

PROPOSITION 7.10. *There are extensions of $\mathcal{R}$-modules*

(1) $\mathcal{A} \hookrightarrow \mathcal{B} \twoheadrightarrow \mathcal{C}$,
(2) $\mathcal{R} \hookrightarrow \mathcal{B}_\infty \twoheadrightarrow \mathcal{C}_\infty$.

Proof. By using Lemmas 7.5, 7.8 and 7.9 we get the following equalities, isomorphisms and short exact sequences, which correspond to the extensions of the statement

$$\frac{\mathcal{R}}{\mathsf{A}\mathcal{R}} \simeq \frac{(\mathsf{I}-\mathsf{A}^t)\mathcal{R}}{(\mathsf{I}-\mathsf{A}^t)\mathsf{A}\mathcal{R}} = \frac{(\mathsf{I}-\mathsf{A}^t)\mathcal{R}}{(\mathsf{I}-\mathsf{A})\mathcal{R}} \hookrightarrow \frac{\mathcal{R}}{(\mathsf{I}-\mathsf{A})\mathcal{R}} \twoheadrightarrow \frac{\mathcal{R}}{(\mathsf{I}-\mathsf{A}^t)\mathcal{R}},$$
$$\mathcal{R} \simeq \frac{\mathcal{R}}{\mathsf{B}\mathcal{R}} \simeq \frac{(\mathsf{I}-\mathsf{B}^t)\mathcal{R}}{(\mathsf{I}-\mathsf{B}^t)\mathsf{B}\mathcal{R}} = \frac{(\mathsf{I}-\mathsf{B}^t)\mathcal{R}}{(\mathsf{I}-\mathsf{B})\mathcal{R}} \hookrightarrow \frac{\mathcal{R}}{(\mathsf{I}-\mathsf{B})\mathcal{R}} \twoheadrightarrow \frac{\mathcal{R}}{(\mathsf{I}-\mathsf{B}^t)\mathcal{R}}.$$

□

The proof of the following proposition is contained in the proof of Proposition 4.3.

PROPOSITION 7.11. *Given a k-vector space V, if* $\dim V < \aleph_0$ *then* $\mathfrak{i}\, V = \mathcal{B}_{\dim V}$*, and* $\mathfrak{i}\, V = \mathcal{B}_\infty$ *if* $\dim V = \aleph_0$.

Corollary 7.12 follows from Proposition 7.11 and Corollary 4.2.

COROLLARY 7.12. *The $\mathcal{R}$-module $\mathcal{B}_d$ is injective for every $d \in \mathbb{N}_\infty$.*

In Lemma 7.13 we show that one can adapt the basis of a countably generated vector space to a decreasing filtration.

LEMMA 7.13. *Let $V_0 \supset V_1 \supset \cdots \supset V_n \supset V_{n+1} \supset \cdots$ be a decreasing sequence of k-vector spaces such that V_0 is the union of an increasing sequence of finite dimensional subspaces $V_0^0 \subset V_0^1 \subset \cdots \subset V_0^n \subset V_0^{n+1} \subset \cdots$, $V_0 = \bigcup_{n \in \mathbb{N}_0} V_0^n$. If we set $V_0^{-1} = 0$, $V_m^n = V_0^n \cap V_m$ $(n+1, m \in \mathbb{N}_0)$, $V_\infty = \bigcap_{n \in \mathbb{N}_0} V_n$ and choose (finite and possibly empty) sets $\{a^l_{nm};\, 1 \leqslant l \leqslant r_{nm}\} \subset V_m^n$ such that the sets*

$$\left\{a^l_{nm} + (V_m^{n-1} + V_{m+1}^n);\, 1 \leqslant l \leqslant r_{nm}\right\}$$

are basis of $V_m^n/(V_m^{n-1}+V_{m+1}^n)$ $(n,m\in\mathbb{N}_0)$*, then given* $m,n,p\in\mathbb{N}_0$ *with* $m\leqslant p$

(1) $\left\{a_{ij}^{l_{ij}}+V_p^n; i\leqslant n, m\leqslant j<p, 1\leqslant l_{ij}\leqslant r_{ij}\right\}$ *is a basis of* V_m^n/V_p^n,

(2) $\left\{a_{ij}^{l_{ij}}+V_\infty; i\leqslant n, m\leqslant j, 1\leqslant l_{ij}\leqslant r_{ij}\right\}$ *is a basis of* $(V_m^n+V_\infty)/V_\infty$,

(3) $\left\{a_{ij}^{l_{ij}}+V_\infty; i\in\mathbb{N}_0, m\leqslant j, 1\leqslant l_{ij}\leqslant r_{ij}\right\}$ *is a basis of* V_m/V_∞.

Proof. Since V_0^n is a finite-dimensional vector space it is artinian and the decreasing sequence $V_0^n\supset V_1^n\supset\cdots\supset V_m^n\subset V_{m+1}^n\supset\cdots$ stabilizes, that is, there exists $M_n\in\mathbb{N}_0$ such that $V_m^n=V_{M_n}^n$ for every $m\geqslant M_n$, in particular $V_{M_n}^n=V_0^n\cap V_\infty$. If we choose for every $n\in\mathbb{N}_0$ the minimum M_n satisfying this condition then $M_n\leqslant M_{n+1}$ since

$$V_{M_{n+1}}^n=V_{M_{n+1}}^n\cap V_{M_{n+1}}^{n+1}=V_0^n\cap V_{M_{n+1}}\cap V_0^{n+1}\cap V_\infty=V_0^n\cap V_\infty=V_{M_n}^n.$$

Notice that (1) is trivial for $m\geqslant M_n$ since $V_m^n=V_p^n=V_{M_n}^n$ and $\{a_{ij}^l; 1\leqslant l\leqslant r_{ij}\}=\emptyset$ whenever $p\geqslant m\geqslant M_n, i\leqslant n$ and $j\geqslant M_n$. Therefore the elements (n,m,p) for which we still have to check (1) lie in the set $S=\{(n,m,p); n\in\mathbb{N}_0, 0\leqslant m\leqslant M_n, p\geqslant m\}$. Let us order this set in the following way

$$(n,m,p)\leqslant(n',m',p')\ \Leftrightarrow\ \begin{cases} n<n' \\ \text{or} \\ n=n' \text{ and } m>m' \\ \text{or} \\ n=n', m=m' \text{ and } p\leqslant p'. \end{cases}$$

One readily checks that this is a well order on S, since the second coordinate has an upper bound (depending on the first one). The minimum of S is $(0,M_0,M_0)$, moreover if $m<M_n$ the element (n,m,m) is the least upper bound of $\{(n,m+1,p); p>m\}$, and given $n>0$ the element (n,M_n,M_n) is the least upper bound of the set $\{(n-1,m,p); m\leqslant M_{n-1}, p\geqslant m\}$. Any other element in S is a successor. We have already checked (1) for the elements $(n,M_n,p)\in S$, moreover, it is trivial for $(n,m,m)\in S$, hence (1) holds for the minimum and all limit elements in S. A generic successor in S has the form $(n,m,p+1)$ for some $p\geqslant m$. Notice that we have already checked (1) for some successors as well, namely for those with $m=M_n$. We are now going to proceed by induction, that is, we shall prove (1) for every successor in S with $m<M_n$ supposing that (1) holds for all the strictly lower elements. We are going to distinguish three cases:

For $(0,m,p+1)$ (1) follows from the exactness of the sequence

$$\frac{V_p^0}{V_{p+1}^0}\hookrightarrow\frac{V_m^0}{V_{p+1}^0}\twoheadrightarrow\frac{V_m^0}{V_p^0},$$

the equality $V_0^{-1}=0$ and the inequality $(0,m,p)<(0,m,p+1)$, and the inequality $(0,p+1,p)<(0,m,p+1)$ if $p<M_0$ or the equalities $V_p^0=V_{p+1}^0=V_{M_0}^0$ and $\{a_{0p}^l;\,1\leqslant l\leqslant r_{0p}\}=\emptyset$ if $p\geqslant M_0$.

For $(n,m,m+1)$ with $n>0$ (1) is a consequence of the exactness of the sequence

$$\frac{V_m^{n-1}+V_{m+1}^n}{V_{m+1}^n}\hookrightarrow\frac{V_m^n}{V_{m+1}^n}\twoheadrightarrow\frac{V_m^n}{V_m^{n-1}+V_{m+1}^n},$$

the obvious isomorphism

$$\frac{V_m^{n-1}}{V_{m+1}^{n-1}}\simeq\frac{V_m^{n-1}+V_{m+1}^n}{V_{m+1}^n},$$

the inequality $(n-1,m,m+1)<(n,m,m+1)$ if $m<M_{n-1}$, or the equalities $V_m^{n-1}=V_{m+1}^{n-1}=V_{M_{n-1}}^{n-1}$ and $\{a_{in}^l;\,1\leqslant l\leqslant r_{in}\}=\emptyset$ if $i\leqslant n-1$ and $m\geqslant M_{n-1}$.

For $(n,m,p+1)$ with $n>0$ and $p>m$ (1) follows from the exactness of the sequence

$$\frac{V_p^n}{V_{p+1}^n}\hookrightarrow\frac{V_m^n}{V_{p+1}^n}\twoheadrightarrow\frac{V_m^n}{V_p^n},$$

the inequality $(n,m,p)<(n,m,p+1)$, and the inequality $(n,p,p+1)<(n,m,p+1)$ if $p<M_n$ or the equalities $V_p^n=V_{p+1}^n=V_{M_n}^n$ and $\{a_{ip}^l;\,1\leqslant l\leqslant r_{ip}\}=\emptyset$ if $i\leqslant n$ and $p\geqslant M_n$.

Once we have seen that (1) holds, (2) is a consequence of (1) for $p=M_n$, the isomorphism $(V_m^n+V_\infty)/V_\infty\simeq V_m^n/V_{M_n}^n$, and the fact that $\{a_{ij}^l;\,1\leqslant l\leqslant r_{ij}\}=\emptyset$ is the empty set for $i\leqslant n$ and $j\geqslant M_n$. Finally (3) follows from (2) and the equality $V_m=\bigcup_{n\in\mathbb{N}_0}V_m^n$.

Proposition 7.14 is an interesting consequence of Lemma 7.13. It does not hold in general when the ground ring is not a field, compare [1].

PROPOSITION 7.14. *The image of a morphism between finitely generated free* $\mathbf{M}_k(\bar{T}_1)$*-modules is finitely generated free.*

Proof. Let $\varphi:k\langle B\rangle_\beta\to k\langle A\rangle_\alpha$ be a morphism in $\mathbf{M}_k(\bar{T}_1)$. We can apply Lemma 7.13 to the k-vector space $V_0=\varphi(k\langle B\rangle)$ and the subspaces $V_n=\varphi(k\langle B_n\rangle)\subset k\langle A\rangle\ (n\in\mathbb{N})$ and $V_0^n=\varphi(k\langle_n B\rangle)\ (n\geqslant 0)$. Moreover, with the notation of that lemma $V_\infty=0$ since for every $n\geqslant 1$ there exists $N_n\geqslant 1$ such that $V_{N_n}\subset k\langle A_n\rangle$ and $\cap_{n\geqslant 1}k\langle A_n\rangle=0$. We define the set $\underline{B}=\{a_{ij}^l;\,i,j\in\mathbb{N}_0,1\leqslant l\leqslant r_{ij}\}$ and the function $\underline{\beta}:\underline{B}\to\mathbb{N}_0\subset[0,+\infty)$ by $\underline{\beta}(a_{nm}^l)=m$. By Lemma 7.13 (3) the set $\underline{B}$ is a basis of V_0 and $\underline{B}_m$ a basis of $V_m\ (m\geqslant 1)$ since V_∞=0. The function β is a height function because the cardinal of $\underline{\beta}^{-1}(m)$ is dim $V_m/V_{m+1}<\aleph_0\ (m\in\mathbb{N}_0)$. Moreover, the inclusion

$k\langle\underline{B}\rangle=\varphi(k\langle B\rangle)\subset k\langle A\rangle$ and the projection $k\langle B\rangle\twoheadrightarrow\varphi(k\langle B\rangle)=k\langle\underline{B}\rangle$ give rise to controlled homomorphisms $k\langle\underline{B}\rangle_{\underline{\beta}}\hookrightarrow k\langle A\rangle_\alpha$ and $k\langle B\rangle_\beta\twoheadrightarrow k\langle\underline{B}\rangle_{\underline{\beta}}$ which are an $\mathbf{M}_k(\bar{T}_1)$-module monomorphism and epimorphism, respectively, and their composition is φ hence $k\langle\underline{B}\rangle_{\underline{\beta}}$ together with these morphisms is the image of φ. □

COROLLARY 7.15. *Any finitely presented* $\mathbf{M}_k(\bar{T}_1)\times$*module is the cokernel of a monomorphism between finitely generated free* $\mathbf{M}_k(\bar{T}_1)$*-modules.*

COROLLARY 7.16. *Finitely presented* $\mathcal{R}$*-modules have projective dimension* $\leqslant 1$.

COROLLARY 7.17. *We have* $\mathrm{Ext}^1(\mathcal{M},\mathcal{C}_d)=0$ *for any f. p.* $\mathcal{R}$*-module* $\mathcal{M}$ *and* $d\in\mathbb{N}_\infty$.

Proof. By Corollary 7.16 the functor $\mathrm{Ext}^1(\mathcal{M},-)$ is right-exact, hence the corollary follows from Proposition 7.10 and Corollary 7.12. □

Now we begin with the lemmas which prove Theorem 7.2.

LEMMA 7.18. *Given any f. p.* $\mathcal{R}$*-module* $\mathcal{M}$*, there exists another f. p.* $\mathcal{R}$*-module* $\mathcal{N}$ *with* $\lambda_{\mathcal{N}}=0$ *such that* $\mathcal{M}\simeq\mathcal{A}_{\lambda_{\mathcal{M}}}\oplus\mathcal{N}$.

Proof. Suppose that $\mathcal{M}=\mathrm{Coker}\,\varphi$ for some $\varphi:k\langle B\rangle_\beta\to k\langle A\rangle_\alpha$ in $\mathbf{M}_k(\bar{T}_1)$. Let us consider the decreasing sequence of k-vector spaces given by $V_0=k\langle A\rangle/\varphi(k\langle B\rangle)$ and

$$V_n=\frac{k\langle A_n\rangle+\varphi(k\langle B\rangle)}{\varphi(k\langle B\rangle)},\quad n\in\mathbb{N}.$$

The vector space V_0 is the union of the following sequence of finite-dimensional k-vector spaces ($n\in\mathbb{N}_0$)

$$V_0^n=\frac{k\langle{}_nA\rangle+\varphi(k\langle B\rangle)}{\varphi(k\langle B\rangle)}.$$

If $\{a^l_{nm};1\leqslant l\leqslant r_{nm}\}\subset V^n_m$ is a set as in Lemma 7.13 we can suppose that $a^l_{nm}=e^l_{nm}+\varphi(k\langle B\rangle)$ for some $e^l_{nm}\in k\langle{}_nA_m\rangle$, here we use the next obvious isomorphism

$$\frac{k\langle{}_nA_m\rangle}{k\langle{}_nA_m\rangle\cap\varphi(k\langle B\rangle)}\simeq\frac{k\langle{}_nA_m\rangle+\varphi(k\langle B\rangle)}{\varphi(k\langle B\rangle)}=V^n_m.$$

We consider the set $C=\{a^{l_{nm}}_{nm}+V_\infty;n,m\in\mathbb{N}_0,1\leqslant l_{nm}\leqslant r_{nm}\}$ and the function $\gamma:C\to\mathbb{N}_0\subset[0,+\infty)$ with $\gamma(a^{l_{nm}}_{nm}+V_\infty)=m$. This function is a height function, since the set $\gamma^{-1}(m)=\{a_{nm}^{l_{nm}}+V_\infty; n\in\mathbb{N}_0, 1\leqslant l_{nm}\leqslant r_{nm}\}$ is bijective with a basis of V_m/V_{m+1} by Lemma 7.13 (3), and we have the following surjection and isomorphisms

$$\frac{k\langle A_m\rangle}{k\langle A_{m+1}\rangle}\twoheadrightarrow\frac{k\langle A_m\rangle}{k\langle A_{m+1}\rangle+[\varphi(k\langle B\rangle)\cap k\langle A_m\rangle]}\simeq\frac{k\langle A_m\rangle+\varphi(k\langle B\rangle)}{k\langle A_{m+1}\rangle+\varphi(k\langle B\rangle)}\simeq\frac{V_m}{V_{m+1}},$$

where $\dim k\langle A_m\rangle/k\langle A_{m+1}\rangle=\text{card }\alpha^{-1}(m)<\aleph_0$. The underlying k-vector space of $k\langle C\rangle_\gamma$ is V_0/V_∞, moreover, the natural projection

$$k\langle A\rangle\twoheadrightarrow\frac{k\langle A\rangle}{\bigcap\limits_{n\geqslant 1}[k\langle A_n\rangle+\varphi(k\langle B\rangle)]}\simeq\frac{V_0}{V_\infty}=k\langle C\rangle \tag{a}$$

gives rise to a $\bar{T}_1$-controlled homomorphism $v_0:k\langle A\rangle_\alpha\to k\langle C\rangle_\gamma$ with $v_0\varphi=0$, hence v_0 induces a morphism $v:\mathcal{M}\to k\langle C\rangle_\gamma$. Furthermore, the section $V_0/V_\infty\hookrightarrow k\langle A\rangle$ which sends $a_{nm}^l+V_\infty$ to e_{nm}^l determines another $\bar{T}_1$-controlled homomorphism $\tau_0:k\langle C\rangle_\gamma\to k\langle A\rangle_\alpha$ with $v_0\tau_0=1$, in particular if $\tau:k\langle C\rangle_\gamma\to\mathcal{M}$ is the morphism induced by τ_0 we have that $v\tau=1$, hence $\mathcal{M}\simeq k\langle C\rangle_\gamma\oplus\mathcal{N}$ where $\mathcal{N}=\text{Coker }\tau$. Notice that the morphism $(\varphi,\tau_0):k\langle B\rangle_\beta\oplus k\langle C\rangle_\gamma\to k\langle A\rangle_\alpha$ is a finite presentation of $\mathcal{N}$, and by (a) we have the following equality and inclusions for every $m\geqslant 1$

$$k\langle A\rangle=\bigcap_{n\geqslant 1}[k\langle A_n\rangle+\varphi(k\langle B\rangle)]\oplus\tau(k\langle C\rangle)\subset k\langle A_m\rangle+\varphi(k\langle B\rangle)+\tau(k\langle C\rangle)\subset k\langle A\rangle,$$

therefore $\lambda_{\mathcal{N}}=0$.

Observe that by Lemmas 7.6 and 7.7 $k\langle C\rangle_\gamma$ is isomorphic to the free $\bar{T}_1$-controlled k-vector space which corresponds to $\mathcal{R}$ provided $\lambda_{\mathcal{M}}=\infty$, and to the direct sum of $\lambda_{\mathcal{M}}$ copies of $\mathcal{A}$ otherwise. □

LEMMA 7.19. *Given a f. p. $\mathcal{R}$-module $\mathcal{M}$ with $\lambda_{\mathcal{M}}=0$, there exists another f. p. $\mathcal{R}$-module $\mathcal{N}$ with $\lambda_{\mathcal{N}}=\mu_{\mathcal{N}}=0$ such that $\mathcal{M}\simeq\mathcal{B}_{\mu_{\mathcal{M}}}\oplus\mathcal{N}$.*

Proof. Suppose that $\mathcal{M}$ is the cokernel of $\varphi:k\langle B\rangle_\beta\to k\langle A\rangle_\alpha$ in $\mathbf{M}_k(\bar{T}_1)$. Since $\lambda_{\mathcal{M}}=0$ we have that $k\langle A\rangle=k\langle A_m\rangle+\varphi(k\langle B\rangle)$ for every $m\in\mathbb{N}$. Let $V_\bullet$ be the inverse system indexed by $\mathbb{N}\times\mathbb{N}$ given by

$$V_{mn}=\frac{k\langle A_m\rangle}{k\langle A_m\rangle\cap\varphi(k\langle B_n\rangle)},$$

and bonding homomorphisms induced by the obvious inclusions of vector spaces. There are inclusions $U_{mn}^{\varphi}\subset V_{mn}$ with quotients

$$\frac{k\langle A_m\rangle}{k\langle A_m\rangle\cap\varphi(k\langle B\rangle)}\simeq\frac{k\langle A_m\rangle+\varphi(k\langle B\rangle)}{\varphi(k\langle B\rangle)}=\frac{k\langle A\rangle}{\varphi(k\langle B\rangle)}.$$

This determines a short exact sequence in the category of pro-vector spaces

$$U^{\varphi}_{\bullet} \hookrightarrow V_{\bullet} \twoheadrightarrow \frac{k\langle A\rangle}{\varphi(k\langle B\rangle)}. \tag{a}$$

Here we regard $k\langle A\rangle/\varphi(k\langle B\rangle)$ as the inverse system indexed by a singleton.

The vector space U^{φ}_{mn} is always finite dimensional, because it is contained in $\varphi(k\langle B\rangle)/\varphi(k\langle B_n\rangle) \simeq \varphi(k\langle_{n-1}B\rangle)$ and ${}_{n-1}B$ is a finite set. Since finite-dimensional vector spaces are artinian it is easy to see that $U^{\varphi}_{\bullet}$ satisfies the Mittag–Leffler property, in particular $\lim^1 U^{\varphi}_{\bullet} = 0$ and hence by (5.b) $\mathrm{Ext}^1(k\langle A\rangle/\varphi(k\langle B\rangle), U^{\varphi}_{\bullet}) = 0$, so the sequence (a) admits a splitting $s : k\langle A\rangle/\varphi(k\langle B\rangle) \hookrightarrow V_{\bullet}$. This splitting is given by splittings $s_{mn} : k\langle A\rangle/\varphi(k\langle B\rangle) \hookrightarrow V_{mn}$ of the natural projections $V_{mn} \twoheadrightarrow k\langle A\rangle/\varphi(k\langle B\rangle)$ which are compatible with the bonding homomorphisms of $V_{\bullet}$.

Let $\tilde{C}$ be a basis of $k\langle A\rangle/\varphi(k\langle B\rangle)$. This basis is either finite $\tilde{C} = \{b_1, \ldots, b_{\mu_{\mathcal{M}}}\}$ if $\mu_{\mathcal{M}} \in \mathbb{N}_0$, or infinite countable $\tilde{C} = \{b_n\}_{n\in\mathbb{N}_0}$ if $\mu_{\mathcal{M}} = \infty$. Moreover, since φ is controlled there exists an increasing sequence of natural numbers $\{l_n\}_{n\geqslant 1}$ with $\varphi(k\langle B_{l_n}\rangle) \subset k\langle A_n\rangle$. We choose elements $b^{n-1}_m \in k\langle A_n\rangle$ and $y^{n-1}_m \in k\langle B_{l_n}\rangle$ such that $b^{n-1}_m + \varphi(k\langle B_{l_n}\rangle) = s_{n,l_n}(b_m) \in V_{n,l_n} = k\langle A_n\rangle/\varphi(k\langle B_{l_n}\rangle)$ and $\varphi(y^{n-1}_m) = b^n_m - b^{n-1}_m$ for every $n \in \mathbb{N}$ and m in the corresponding range. Furthermore, we define the sets ${}^nC \subset k\langle A_{n+1}\rangle$ and C in the following way: ${}^nC = \{b^n_1, \ldots, b^n_{\mu_{\mathcal{M}}}\}$ and $C = \coprod_{n\in\mathbb{N}_0} {}^nC =$ if $\mu_{\mathcal{M}} \in \mathbb{N}_0$, and ${}^nC = \{b^n_0, \ldots, b^n_n\} \cup \{b^m_m; m > n\}$ and $C = \bigcup_{n\in\mathbb{N}_0} {}^nC = \{b^n_m; n \geqslant m \geqslant 0\}$ if $\mu_{\mathcal{M}} = \infty$. Let $\gamma : C \to \mathbb{N}_0 \subset [0, +\infty)$ be the height function given by $\gamma(b^n_m) = n$ and ψ the endomorphism of $k\langle C\rangle_\gamma$ given by $\psi(b^n_m) = b^{n+1}_m - b^n_m$.

One readily checks that Coker $\psi = \mathfrak{i}(k\langle A\rangle/\varphi(k\langle B\rangle))$ and the natural projection $k\langle C\rangle_\gamma \twoheadrightarrow \mathfrak{i}(k\langle A\rangle/\varphi(k\langle B\rangle))$ is given by the homomorphism $p_1 : k\langle C\rangle \to k\langle A\rangle/\varphi(k\langle B\rangle) = k\langle\tilde{C}\rangle$ defined by $p_1(b^n_m) = b_m$. For this one uses the finite presentations constructed in the proof of Proposition 4.3 and, if $\mu_{\mathcal{M}} = \infty$, the bijection $\mathbb{N}_0 \approx C$ which sends $m \in \mathbb{N}_0$, with $n(n-1)/2 \leqslant m < (n+1)n/2$ for some $n \in \mathbb{N}_0$, to $b^{n-1}_{m-\frac{n(n-1)}{2}}$. Moreover, by Proposition 7.11 Coker $\psi = \mathcal{B}_{\mu_{\mathcal{M}}}$.

The homomorphism $\tau_0 : k\langle C\rangle \to k\langle A\rangle$ induced by the inclusions ${}^nC \subset k\langle A_{n+1}\rangle \subset k\langle A\rangle$ determines a controlled homomorphism $\tau_0 : k\langle C\rangle_\gamma \to k\langle A\rangle_\alpha$. Moreover, the homomorphism $\tau_1 : k\langle C\rangle \to k\langle B\rangle$ given by $\tau_1(b^n_m) = y^n_m$ defines a controlled homomorphism $\tau_1 : k\langle C\rangle_\gamma \to k\langle B\rangle_\beta$ with $\varphi\tau_1 = \tau_0\psi$, hence τ_0 gives rise to a $\mathbf{M}_k(\bar{T}_1)$-module morphism $\tau : \mathcal{B}_{\mu_{\mathcal{M}}} \to \mathcal{M}$.

Let us check that τ is a monomorphism. Given a free $\bar{T}_1$-controlled vector space $k\langle D\rangle_\phi$ Yoneda's lemma yields a natural identification $\mathrm{Hom}_{\mathcal{R}}(k\langle D\rangle_\phi, \mathfrak{i}(k\langle A\rangle/\varphi(k\langle B\rangle))) = \mathrm{Hom}_k(k\langle D\rangle, k\langle A\rangle/\varphi(k\langle B\rangle))$. This identification carries a morphism $v : k\langle D\rangle_\phi \to \mathfrak{i}(k\langle A\rangle/\varphi(k\langle B\rangle))$ represented by $v_0 : k\langle D\rangle_\phi \to k\langle C\rangle_\gamma$ to the vector space homomorphism $p_1 v_0$. If $p_2 : k\langle A\rangle \twoheadrightarrow k\langle A\rangle/\varphi(k\langle B\rangle)$ is the natural projection then $p_2\varphi = 0$ and $p_1 = p_2\tau_0$. Moreover $\tau v = 0$ if and only if $\tau_0 v_0 = \varphi\eta$ for some controlled homomorphism

$\eta: k\langle D\rangle_\phi \to k\langle B\rangle_\beta$, so in this case $p_1 v_0 = p_2\tau_0 v_0 = p_2\varphi\eta = 0$, that is, $v = 0$, therefore τ is a monomorphism.

By Corollary 7.12 if $\mathcal{N} = \operatorname{Coker}\tau$ then $\mathcal{M} \simeq \mathcal{B}_{\mu_{\mathcal{M}}} \oplus \mathcal{N}$. The morphism $(\varphi, \tau_0): k\langle B\rangle_\beta \oplus k\langle C\rangle_\gamma \to k\langle A\rangle_\alpha$ is a finite presentation of $\mathcal{N}$ and by construction $\varphi(k\langle B\rangle) + \tau_0(k\langle C\rangle) = k\langle A\rangle$, it is $\lambda_{\mathcal{N}} = \mu_{\mathcal{N}} = 0$. □

LEMMA 7.20. *Let $\mathcal{M}$ be a f. p. $\mathcal{R}$-module with $\lambda_{\mathcal{M}} = \mu_{\mathcal{M}} = 0$ then there exists another one $\mathcal{N}$ with $\lambda_{\mathcal{N}} = \mu_{\mathcal{N}} = \nu_{\mathcal{N}} = 0$ such that $\mathcal{M} \simeq \mathcal{C}_{\nu_{\mathcal{M}}} \oplus \mathcal{N}$.*

In the proof of this lemma we shall use the following.

LEMMA 7.21. *Let $\{d_n\}_{n\in\mathbb{N}}$ be an increasing sequence of integers with $\lim\limits_{n\to\infty} d_n = \infty$. Consider the set $A = \{(n,m); n \in \mathbb{N}, m \leqslant d_n\} \subset \mathbb{N}\times\mathbb{N}$, the height function $\alpha: A \to \mathbb{N}_0 \subset [0,+\infty)$ with $\alpha(n,m) = n$, and the endomorphism φ of $k\langle A\rangle_\alpha$ with $\varphi(n,m) = (n,m)$ if $n = 1$ or $n > 1$ and $m > d_{n-1}$, and $\varphi(n,m) = (n,m) - (n-1,m)$ otherwise. Then* $\operatorname{Coker}\varphi$ *is isomorphic to $\mathcal{C}_\infty$.*

Proof. Consider the infinite countable subsets

$$A_1 = \{(1,m); 1 \leqslant m \leqslant d_1\} \cup \{(n,m); n > 1, d_{n-1} < m \leqslant d_n\} \subset A,$$

$A_2 = A - A_1$, $B_1 = \{(n+1)n/2\}_{n\in\mathbb{N}_0} \subset \mathbb{N}_0$ and $B_2 = \mathbb{N}_0 - B_1$. The lexicographic order from the left on A is a well order without limit elements, since the second coordinate of an element $(n,m) \in A$ is bounded by d_n, hence the restriction of this order to the subsets A_1 and A_2 induces enumerations $A_1 = \{e_1^n\}_{n\in\mathbb{N}_0}$ and $A_2 = \{e_2^n\}_{n\in\mathbb{N}_0}$. Similarly the usual order in $\mathbb{N}_0$ induces enumerations in the subsets $B_1 = \{f_1^n\}_{n\in\mathbb{N}_0}$ and $B_2 = \{f_2^n\}_{n\in\mathbb{N}_0}$. Now the theorem follows from the bijection $\mathbb{N}_0 \approx A$ which sends f_i^n to e_i^n $(i = 1,2; n \in \mathbb{N}_0)$. □

Proof of Lemma 7.20. If $\mathcal{M} = \operatorname{Coker}[\varphi: k\langle B\rangle_\beta \to k\langle A\rangle_\alpha]$ the equalities $\lambda_{\mathcal{M}} = \mu_{\mathcal{M}} = 0$ are equivalent to $\varphi(k\langle B\rangle) = k\langle A\rangle$. Let $\phi: \lim U_\bullet^\varphi \to U_\bullet^\varphi$ be the canonical pro-morphism. This pro-morphism is given by vector space homomorphisms $\phi_{mn}: \lim U_\bullet^\varphi \to U_{mn}^\varphi$ compatible with the bonding homomorphisms of $U_\bullet^\varphi$. Since φ is controlled there is an increasing sequence of natural numbers $\{m_n\}_{n\geqslant 1}$ such that $\varphi(k\langle B_{mn}\rangle) \subset k\langle A_n\rangle$.

If $\nu_{\mathcal{M}} \in \mathbb{N}_0$ and $\{a_1, \ldots, a_{\nu_{\mathcal{M}}}\}$ is a basis of $\lim U_\bullet^\varphi$, we define ${}^nC = \{a_1^n, \ldots, a_{\nu_{\mathcal{M}}}^n\} \subset k\langle A_n\rangle$ as a set such that $\phi_{n,m_n}(a_i) = a_i^n + \varphi(k\langle B_{m_n}\rangle)$ $(1 \leqslant i \leqslant \nu_{\mathcal{M}})$, and choose elements $y_i^n \in k\langle B_{m_{n-1}}\rangle$ if $n > 1$ and $y_i^1 \in k\langle B\rangle$ with $a_i^n - a_i^{n-1} = \varphi(y_i^n)$ $(n > 1)$ and $a_i^1 = \varphi(y_i^1)$. If $\nu_{\mathcal{M}} = \infty$ we take ${}^nC = \{a_i^n\}_{i=1}^{d_n} \subset k\langle A_n\rangle$ such that $\{a_i^n + \varphi(k\langle B_{m_n}\rangle)\}_{i=1}^{d_n}$ is a basis of $\phi_{n,m_n}(\lim U_\bullet^\varphi)$, here we use that $U_\bullet^\varphi$ is an inverse system of finite-dimensional vector spaces, compare the proof of Lemma 7.19. The bonding homomorphisms of $U_\bullet^\varphi$ induce surjections $\phi_{n+1,m_{n+1}}(\lim U_\bullet^\varphi) \twoheadrightarrow \phi_{n,m_n}(\lim U_\bullet^\varphi)$, hence $d_n \leqslant d_{n+1}$ and we can suppose without loss of generality that there exist $y_i^n \in k\langle B_{m_{n-1}}\rangle$ $(n>1)$ and $y_i^1 \in k\langle B\rangle$ such that $a_i^n - a_i^{n-1} = \varphi(y_i^n)$ if $n>1$ and $i \leqslant d_{n-1}$, and $a_i^n = \varphi(y_i^n)$ if $n>1$ and $d_{n-1} < i \leqslant d_n$ or $n=1$ and $i \leqslant d_1$.

We define the height function $\gamma : C = \coprod_{n\geqslant 1} {}^nC \to \mathbb{N}_0 \subset [0,+\infty)$ as $\gamma(a_i^n) = n$, and the controlled homomorphisms $\tau_0 : k\langle C\rangle_\gamma \to k\langle A\rangle_\alpha$, $\tau_1 : k\langle C\rangle_\gamma \to k\langle B\rangle_\beta$, $\psi : k\langle C\rangle_\gamma \to k\langle C\rangle_\gamma$ by $\tau_0(a_i^n) = a_i^n$, $\tau_1(a_i^n) = y_i^n$, and $\psi(a_i^n) = a_i^n - a_i^{n-1}$ if $n>1$, and $\nu_{\mathcal{M}} \in \mathbb{N}_0$ or $\nu_{\mathcal{M}} = \infty$ and $i \leqslant d_{n-1}$, and $\psi(a_i^n) = a_i^n$ otherwise. One can readily check, by using the bijection $\mathbb{N} \approx \mathbb{N}_0 : n \mapsto n-1$ if $\nu_{\mathcal{M}} \in \mathbb{N}_0$ or Lemma 7.21 if $\nu_{\mathcal{M}} = \infty$, that $\operatorname{Coker}\psi \simeq \mathcal{C}_{\nu_{\mathcal{M}}}$. Moreover, $(\tau_1,\tau_0) : \psi \to \varphi$ is a morphism in $\mathbf{pair}(\mathbf{M}_k(\bar{T}_1))$ which induces an $\mathcal{R}$-module morphism $\tau : \mathcal{C}_{\nu_{\mathcal{M}}} \to \mathcal{M}$.

In order to check that τ is a monomorphism of $\mathbf{M}_k(\bar{T}_1)$-modules we first prove that ϕ above is a monomorphism of pro-vector spaces if $\nu_{\mathcal{M}} \in \mathbb{N}_0$. In this case $\operatorname{Ker}\phi$ is an inverse system of finite-dimensional vector spaces, in particular it satisfies the Mittag–Leffler property. If we apply the left-exact functor lim to the exact sequence of pro-vector spaces

$$\operatorname{Ker}\phi \hookrightarrow \lim U^\varphi_\bullet \xrightarrow{\phi} U^\varphi_\bullet$$

we get another one

$$\lim \operatorname{Ker}\phi \hookrightarrow \lim \lim U^\varphi_\bullet \xrightarrow{=} \lim U^\varphi_\bullet,$$

so $\lim \operatorname{Ker}\phi = 0$ and hence $\operatorname{Ker}\phi = 0$ by [12], II.6.2 Lemma 2, therefore ϕ is a monomorphism. In particular there exists $N \in \mathbb{N}$ big enough such that ϕ_{n,m_n} is an injective homomorphism for every $n \geqslant N$. We set $N=1$ if $\nu_{\mathcal{M}} = \infty$. Now it is easy to see that the injection $\psi_n : k\langle C_{n+1}\rangle \hookrightarrow k\langle C_n\rangle$ given by the restriction of ψ is the kernel of the next composite $(n \geqslant N)$

$$k\langle C_n\rangle \stackrel{\tau_0}{\to} k\langle A_n\rangle \twoheadrightarrow \frac{k\langle A_n\rangle}{\varphi(k\langle B_{mn}\rangle)} = U^\varphi_{n,m_n}.$$

Any morphism $v : k\langle D\rangle_\chi \to \mathcal{C}_{\nu_{\mathcal{M}}}$ is represented by a controlled homomorphism $v_0 : k\langle D\rangle_\chi \to k\langle C\rangle_\gamma$. Suppose that $\tau v = 0$. This means that there exists another controlled homomorphism $\eta : k\langle D\rangle_\chi \to k\langle B\rangle_\beta$ with $\tau_0 v_0 = \varphi\eta$. By the alternative characterization of controlled homomorphisms given in Section 3.1 we see that there exists an increasing sequence of natural numbers $\{p_n\}_{n\geqslant 1}$ such that $v_0(k\langle D_{p_n}\rangle) \subset k\langle C_n\rangle$ and $\eta(k\langle D_{p_n}\rangle) \subset k\langle B_{m_n}\rangle$. Hence if $n \geqslant N$ then there exists a unique homomorphism $\sigma_n : k\langle D_{p_n}\rangle \to k\langle C_{n+1}\rangle$ such that $\psi_n\sigma_n : k\langle D_{p_n}\rangle \to k\langle C_n\rangle$ is the restriction of v_0. If $\sigma : k\langle {}_{p_{N-1}}D\rangle \to k\langle C\rangle$ is any homomorphism such that $\psi\sigma'$ coincides with the restriction of v_0 to $k\langle {}_{p_{N-1}}D\rangle$ we define the controlled homomorphism $\sigma : k\langle D\rangle_\chi \to k\langle C\rangle$ by $\sigma(d) = \sigma_n(d)$ if $d \in {}_{p_{n+1}-1}D_{p_n} (n \geqslant N)$ and $\sigma(d) = \sigma'(d)$ if $d \in {}_{p_{N-1}}D$. This controlled homomorphism satisfies $\psi\sigma = v_0$ therefore $v = 0$ and hence τ is a monomorphism.

Since τ is a monomorphism if we define $\mathcal{N} = \mathrm{Coker}\,\tau$ we get by Corollary 7.17 that $\mathcal{M} \simeq \mathcal{C}_{\nu_{\mathcal{M}}} \oplus \mathcal{N}$. Now one can check that $\lambda_{\mathcal{N}} = \mu_{\mathcal{N}} = \nu_{\mathcal{N}} = 0$ by using that $\mathcal{N} = \mathrm{Coker}\,[(\varphi, \tau_0) : k\langle B\rangle_\beta \oplus k\langle C\rangle_\gamma \to k\langle A\rangle_\alpha]$. □

LEMMA 7.22. *If $\mathcal{M}$ is a f. p. $\mathcal{R}$-module with $\lambda_{\mathcal{M}} = \mu_{\mathcal{M}} = \nu_{\mathcal{M}} = 0$ then $\mathcal{M} = 0$.*

Proof. If $\mathcal{M} = \mathrm{Coker}[\varphi : k\langle B\rangle_\beta \to k\langle A\rangle_\alpha]$ the conditions of the statement are equivalent to $\varphi(k\langle B\rangle) = k\langle A\rangle$ and $\lim U^\varphi_\bullet = 0$. In the proof of Lemma 7.19 we checked that $U^\varphi_\bullet$ satisfies the Mittag–Leffler property, hence $U^\varphi_\bullet = 0$ is a trivial pro-vector space by [12], II.6.2 Lemma 2. This means that if $\{m_n\}_{n\geqslant 1}$ is an increasing sequence such that $\varphi(k\langle B_{m_n}\rangle) \subset k\langle A_n\rangle$ (see Section 3.1) then there exists another increasing sequence $\{p_n\}_{n\geqslant 1}$ such that the next bonding homomorphisms

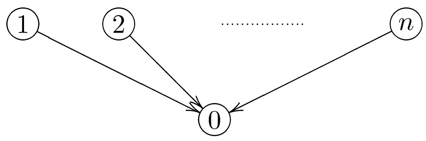
$$U^\varphi_{p_{n+1}, m_{p_{n+1}}} = \frac{k\langle A_{p_{n+1}}\rangle}{\varphi(k\langle B_{m_{p_{n+1}}}\rangle)} \xrightarrow{0} \frac{k\langle A_{p_n}\rangle}{\varphi(k\langle B_{m_{p_n}}\rangle)} = U^\varphi_{p_n, m_{p_n}}$$

are trivial, that is, $k\langle A_{p_{n+1}}\rangle \subset \varphi(k\langle B_{m_{p_n}}\rangle)$. Hence we can define a controlled homomorphism $\psi : k\langle A\rangle_\alpha \to k\langle B\rangle_\beta$ sending $a \in {}_{p_{n+2}-1}A_{p_{n+1}} (n \geqslant 1)$ to any element $b \in B_{m_{p_n}}$ such that $\varphi(b) = a$, and if $a \in {}_{p_2}A$ we take any $\psi(a) = b \in k\langle B\rangle$ such that $\varphi(b) = a$. This morphism satisfies $\varphi\psi = 1$ hence φ is an epimorphism and $\mathcal{M} = \mathrm{Coker}\,\varphi = 0$. □

8. Representations of the *n*-Subspace Quiver

In this section we recall well-known facts one the representation theory of the n-subspace quiver. We also define and briefly study a new class of n-subspaces which we call rigid. These results will play an important role in our treatment of the representation theory of the algebras $k(\boldsymbol{n})$, see Sections 9 and 10.

The *n-subspace quiver* Q_n is the following directed graph



Fixed any field k, a *representation* $\underline{V}$ of Q_n is a diagram of k-vector spaces indexed by Q_n, that is, $n+1$ vector spaces $V_0, V_1, \ldots, V_n$ together with homomorphisms $V_i \to V_0$ $(1 \leqslant i \leqslant n)$. Morphisms of representations are commutative diagrams. The category $\mathbf{rep}_{Q_n}$ of representations of Q_n is an abelian category. It is equivalent to the category of kQ_n-modules, where kQ_n is the path algebra of Q_n, whose dimension is $\dim kQ_n = 2n+1$. A representation is said to be finite-dimensional provided V_i is a finite-dimensional vector space for every $0 \leqslant i \leqslant n$. Finitely presented (or equivalently finite-dimensional) kQ_n-modules correspond to finite-dimensional representations under the equivalence above and indecomposable representations are finite-dimensional, hence by the classical Krull–Schmidt theorem the monoid $\mathrm{Iso}(\mathbf{rep}^{\mathrm{fin}}_{Q_n})$ of isomorphisms classes of finite-dimensional representations of Q_n is the free abelian monoid generated by the (isomorphism classes of) indecomposable representations. The representation type of the quiver Q_n is that of its path algebra.

An *n-subspace* $\underline{V}$ is a representation of Q_n such that the homomorphisms $V_i \to V_0$ are inclusions of subspaces $V_i \subset V_0$ $(1 \leqslant i \leqslant n)$. The category $\mathbf{sub}_n$ (resp. $\mathbf{sub}^{\mathrm{fin}}_n$) of (finite-dimensional) n-subspaces is a full additive (small) subcategory of $\mathbf{rep}_{Q_n}$ (resp. $\mathbf{rep}^{\mathrm{fin}}_{Q_n}$). In fact direct summands in $\mathbf{rep}_{Q_n}$ of n-subspaces are also n-subspaces, hence

PROPOSITION 8.1. $\mathrm{Iso}(\mathbf{sub}^{\mathrm{fin}}_n)$ *is the free abelian monoid generated by the isomorphism classes of indecomposable n-subspaces.*

Up to isomorphism there are just n indecomposable representations of Q_n which are not n-subspaces, namely those with $V_i = k$ for some $1 \leqslant i \leqslant n$ and $V_j = 0$ if $j \neq i$, therefore, we have the following.

PROPOSITION 8.2. *The representation type of* $\mathbf{sub}^{\mathrm{fin}}_n$ *is the same as the representation type of the n-subspace quiver.*

We say that an n-subspace $\underline{V}$ is *rigid* provided $V_i \subset \sum_{j \neq 0,i} V_j$ $(1 \leqslant i \leqslant n)$ and $V_0 = \sum_{i=1}^{n} V_i$. As above, the category $\mathbf{sub}^{\mathrm{rig}}_n$ (resp. $\mathbf{sub}^{\mathrm{fr}}_n$) of (finite-dimensional) rigid n-subspaces is an additive (small) subcategory of $\mathbf{sub}_n$ (resp. $\mathbf{sub}^{\mathrm{fin}}_n$) and direct summands of rigid n-subspaces are also rigid, so we get the following result.

PROPOSITION 8.3. $\mathrm{Iso}(\mathbf{sub}^{\mathrm{fr}}_n)$ *is the free abelian monoid generated by the isomorphism classes of indecomposable rigid n-subspaces.*

There is an additive 'rigidification' functor

$$\mathbf{sub}_n \to \mathbf{sub}^{\mathrm{rig}}_n : \underline{V} \mapsto \underline{V}^{\mathrm{vig}}$$

given by $V_i^{\mathrm{rig}} = V_i \cap (\sum_{j \neq 0,i} V_j)$ $(1 \leqslant i \leqslant n)$ and $V_0^{\mathrm{rig}} = \sum_{i=1}^{n} V_i^{\mathrm{rig}}$, which is right-adjoint to the inclusion $\mathbf{sub}^{\mathrm{rig}}_n \subset \mathbf{sub}_n$ and preserves finite-dimensional objects. The unit of this adjunction is the obvious natural inclusion $\underline{V}^{\mathrm{rig}} \subset \underline{V}$, which is an equality if and only if $\underline{V}$ is already rigid.

In order to determine indecomposable rigid n-subspaces we consider the full inclusions of additive categories $\mathbb{F}^i : \mathbf{sub}_1 \to \mathbf{sub}_n$ $(1 \leqslant i \leqslant n)$ sending a 1-subspace $\underline{W}$ to the n-subspace $\mathbb{F}^i \underline{W} = {}^i\underline{W}$ with ${}^iW_0 = W_0, {}^iW_i = W_1$ and ${}^iW_j = 0$ otherwise.

PROPOSITION 8.4. *The natural inclusion $\underline{V}^{\mathrm{rig}} \subset \underline{V}$ admits a (not natural) retraction in $\mathbf{sub}_n$. More precisely, there exist 1-subspaces $\underline{V}^i$ $(1 \leqslant i \leqslant n)$ and an isomorphism $\underline{V} \simeq (\bigoplus_{i=1}^n \mathbb{F}^i \underline{V}^i) \oplus \underline{V}^{\mathrm{rig}}$ such that the natural inclusion $\underline{V}^{\mathrm{rig}} \subset \underline{V}$ corresponds to the inclusion of the direct summand.*

Proof. By using the definition of $\underline{V}^{\mathrm{rig}}$ we see that there is a short exact sequence of vector spaces

$$\bigoplus_{i=1}^n \frac{V_i}{V_i^{\mathrm{rig}}} \hookrightarrow \frac{V_0}{V_0^{\mathrm{rig}}} \twoheadrightarrow \frac{V_0}{\sum_{i=1}^n V_i}.$$

We define the 1-subspaces $\underline{V}^i$ $(1 \leqslant i \leqslant n)$ as $V_0^1 = (V_0/\sum_{i=1}^n V_i) \oplus (V_1/V_1^{\mathrm{rig}})$, $V_1^1 = V_1/V_1^{\mathrm{rig}}$ and $V_0^i = V_1^i = V_i/V_i^{\mathrm{rig}}$ if $1 \leqslant i \leqslant n$. Now the isomorphism of the statement follows from the previous exact sequence. $\square$

By this proposition an indecomposable n-subspace is rigid unless it is isomorphic to $\mathbb{F}^i \underline{V}$ for some indecomposable 1-subspace $\underline{V}$ and $1 \leqslant i \leqslant n$. It is known that such $\underline{V}$ must be either $k \to k$ or $0 \to k$, so there are just $2n$ indecomposable n-subspaces which are not rigid, and $3n$ indecomposable representations of Q_n which are not rigid n-subspaces, in particular, we have the following.

PROPOSITION 8.5. *The category $\mathbf{sub}_n^{\mathrm{fr}}$ has the same representation type as the n-subspace quiver.*

Remark 8.6. The representation type of the n-subspace quiver is well-known. It is finite for $n < 4$, tame for $n = 4$ and wild if $n > 4$ (see [8,15]).

In [8] the finite sets of indecomposable representations of Q_n are described for $n < 4$ hence discarding the $3n$ indecomposable representations previously described which are not rigid n-subspaces we get the next result.

PROPOSITION 8.7. *The following are complete lists of (representatives of the isomorphism classes of) indecomposable rigid n-subspaces for $n < 4$*

- $n = 1$, *none*,
- $n = 2$, $\underline{V}^{(2,1)} = (k \to k \leftarrow k)$,
- $n = 3$,

$$\underline{V}^{(3,1)} = \begin{pmatrix} & k & \\ & \downarrow & \\ k \to & k & \leftarrow 0 \end{pmatrix}, \qquad \underline{V}^{(3,2)} = \begin{pmatrix} & 0 & \\ & \downarrow & \\ k \to & k & \leftarrow k \end{pmatrix},$$

$$\underline{V}^{(3,3)} = \begin{pmatrix} & k & \\ & \downarrow & \\ 0 \to & k & \leftarrow k \end{pmatrix}, \qquad \underline{V}^{(3,4)} = \begin{pmatrix} & k & \\ & \downarrow & \\ k \to & k & \leftarrow k \end{pmatrix},$$

$$\underline{V}^{(3,5)} = \begin{pmatrix} & k\langle x+y\rangle & \\ & \downarrow & \\ k\langle x\rangle \to & k\langle x, y\rangle & \leftarrow k\langle y\rangle \end{pmatrix}.$$

Remark 8.8. In [15] there is a (not finite) list of indecomposable representations of Q_4 . We do not include the list here because it is quite tedious to describe, however the interested reader can easily find and remove the 12 indecomposable representations of Q_4 which are not rigid 4-subspaces, obtaining in this way a complete list of indecomposable rigid 4-subspaces .

9. Finitely Presented $k(\boldsymbol{n})$-Modules and Finite-dimensional $\boldsymbol{n}$-Subspaces

Once we have solved in Section 7 the decomposition problem for f. p. $k(\boldsymbol{1})$- modules in this section we relate the representation theory of $k(\boldsymbol{n})$ and the n-subspace quiver. This will somehow allow us to decompose the representation problem for f. p. $k(\boldsymbol{n})$-modules into the problem for finite-dimensional rigid n-subspaces and n times the problem for f. p. $k(\boldsymbol{1})$-modules, (see Theorem 10.1).

Given an n-subspace $\underline{V}$ we define $\mathbb{M}\underline{V} : \mathbf{M}_k(\bar{T}_n)^{op} \to \mathbf{Ab}$ as the additive functor which sends an object $k\langle A\rangle_\alpha$ to the vector subspace $\mathbb{M}\underline{V}(k\langle A\rangle_\alpha) \subset \mathrm{Hom}_k(k\langle A\rangle, V_0)$ formed by the homomorphisms $\phi : k\langle A\rangle \to V_0$ such that there exists $M \geqslant 1$ depending on ϕ satisfying $\phi(A^i_M) \subset V_i$ $(1 \leqslant i \leqslant n)$. This construction defines an exact full inclusion of additive categories

$$\mathbb{M} : \mathbf{sub}_n \to \mathbf{mod}(\mathbf{M}_k(\bar{T}_n)). \tag{9.a}$$

PROPOSITION 9.1. *If $\underline{V}$ is a finite-dimensional n-subspace then the $\mathbf{M}_k(\bar{T}_n)$-module $\mathbb{M}\underline{V}$ is finitely presented.*

Proof. Let $\{w_1, \dots , w_d\}$ be a basis of $V_0, \{w^i_1, \dots , w^i_{d_i}\}$ a basis of V_i $(1 \leqslant i \leqslant n)$, and $\phi_i : V_i \to V_0$ the inclusion. We define the sets $D = \{{}_m w^i_1, \dots , {}_m w^i_{d_i}; 1 \leqslant i \leqslant n, m \geqslant 1\}$ and $C = D \sqcup \{w_1, \dots , w_d\}$, and the height functions $\gamma : C \to T^0_n$ and $\delta : D \to T^0_n$ with $\gamma(w_j) = v_0$ $(1 \leqslant j \leqslant d)$ and $\gamma({}_m w^i_j) = \delta({}_m w^i_j) = v^i_m$ $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant d_i, m \geqslant 1)$. Let $\rho : k\langle D\rangle_\delta \to k\langle C\rangle_\gamma$ be the controlled homomorphism defined as $\rho({}_m w^i_j) = {}_m w^i_j - {}_{m-1} w^i_j$ if $m > 1$ and $\rho({}_1 w^i_j) = {}_1 w^i_j - \phi_i(w^i_j)$ otherwise, and $p : k\langle C\rangle_\gamma \to \mathbb{M}\underline{V}$ the

$\mathbf{M}_k(\bar{T}_n)$-module morphism determined by the k-vector space homomorphism $p_0 : k\langle C\rangle \to V_0$ with $p_0(w_j) = w_j$ and $p_0({}_m w_j^i) = \phi_i(w_j^i)$. Here we use that $\mathrm{Hom}(k\langle C\rangle_\gamma, \mathbb{M}\underline{V}) = \mathbb{M}\underline{V}(k\langle C\rangle_\gamma)$ by Yoneda's lemma. Now it is immediate to check that $\mathbb{M}\underline{V}$ = Coker ρ and p is the natural projection. □

One can check by using the finite presentation constructed in the proof of Proposition 9.1 that follows.

COROLLARY 9.2. *If* $\underline{V}$ *is a finite-dimensional rigid n-subspace then* $\Phi_n([\mathbb{M}\underline{V}]) = 0$.

By Proposition 9.1 the additive functor in (9.a) restricts to a functor $\mathbb{M} : \mathbf{sub}_n^{\mathrm{fin}} \to \mathbf{fp}(\mathbf{M}_k(\bar{T}_n))$. Now we are going to construct a functor in the opposite direction. For this if $\varphi : k\langle B\rangle_\beta \to k\langle A\rangle_\alpha$ is a morphism in $\mathbf{M}_k(\bar{T}_n)$ we define the k-vector spaces $(1 \leqslant i \leqslant n)$

$$W_i^\varphi = \frac{\bigcap_{m\geqslant 1}\left\{[k\langle A_m^i\rangle + \varphi(k\langle B\rangle)] \cap \left[\sum_{j\neq i} k\langle A_m^j\rangle + \varphi(k\langle B\rangle)\right]\right\}}{\varphi(k\langle B\rangle)}.$$

Here we use the notation introduced in Section 3.1. This notation will be used along this section without any further mention.

PROPOSITION 9.3. *The vector space* W_i^φ *is finite-dimensional* $(1 \leqslant i \leqslant n)$.

This proposition is an immediate consequence of the following lemma.

LEMMA 9.4. *For any* $m \geqslant 1$

$$\dim \frac{[k\langle A_m^i\rangle + \varphi(k\langle B\rangle)] \cap \left[\sum_{j\neq i} k\langle A_m^j\rangle + \varphi(k\langle B\rangle)\right]}{\varphi(k\langle B\rangle)} < \aleph_0.$$

Proof. One readily checks that

$$\begin{aligned}&[k\langle A_m^i\rangle + \varphi(k\langle B\rangle)] \cap \left[\sum_{j\neq i} k\langle A_m^j\rangle + \varphi(k\langle B\rangle)\right]\\&\quad = k\langle A_m^i\rangle \cap \left[\sum_{j\neq i} k\langle A_m^j\rangle + \varphi(k\langle B\rangle)\right] + \varphi(k\langle B\rangle),\end{aligned}$$

therefore

$$\frac{[k\langle A_m^i\rangle+\varphi(k\langle B\rangle)]\cap\left[\sum_{j\neq i}k\langle A_m^j\rangle+\varphi(k\langle B\rangle)\right]}{\varphi(k\langle B\rangle)}$$
$$\simeq\frac{k\langle A_m^i\rangle\cap\left[\sum_{j\neq i}k\langle A_m^j\rangle+\varphi(k\langle B\rangle)\right]}{k\langle A_m^i\rangle\cap\varphi(k\langle B\rangle)}. \tag{a}$$

Since φ is a controlled homomorphism there exists $M\geqslant 1$ such that $\varphi(B_M^i)\subset k\langle A_m^i\rangle$ $(1\leqslant i\leqslant n)$, hence

$$\sum_{j\neq i}k\langle A_m^j\rangle+\varphi(k\langle B\rangle)=\sum_{j\neq i}k\langle A_m^j\rangle+\varphi(k\langle {}_{M-1}B\rangle)+\varphi(k\langle B_M^i\rangle). \tag{b}$$

The set ${}_{M-1}B$ is finite, hence there exists $N\geqslant 0$ big enough with $\varphi({}_{M-1}B)\subset k\langle {}_NA\rangle$. Let us check that the next homomorphism induced by the inclusion ${}_NA_m^i\subset A_m^i$ is an isomorphism

$$\frac{k\langle {}_NA_m^i\rangle\cap\left[\sum_{j\neq i}k\langle A_m^j\rangle+\varphi(k\langle B\rangle)\right]}{k\langle {}_NA_m^i\rangle\cap\varphi(k\langle B\rangle)}\to\frac{k\langle A_m^i\rangle\cap\left[\sum_{j\neq i}k\langle A_m^j\rangle+\varphi(k\langle B\rangle)\right]}{k\langle A_m^i\rangle\cap\varphi(k\langle B\rangle)}. \tag{c}$$

The injectivity is obvious. Now by (b) an arbitrary element in the range of (c) is represented by an element $a_i\in k\langle A_m^i\rangle$ such that there are $a_j\in k\langle A_m^j\rangle$ $(j\neq i), a_i'\in k\langle A_m^i\rangle\cap\varphi(k\langle B\rangle), b\in\varphi(k\langle {}_{M-1}B\rangle)$, and $b_i\in\varphi(k\langle B_M^i\rangle)$ such that

$$\sum_{j\neq i}a_j+b+b_i=a_i+a_i',$$

but $a_i+a_i'-b_i\in k\langle A_m^i\rangle, (\oplus_{j=1}^n k\langle A_m^j\rangle)\cap k\langle {}_NA\rangle=\oplus_{j=1}^n k\langle {}_NA_m^j\rangle$, and $\sum_{j\neq i}a_j-(a_i+a_i'-b_i)=b\in k\langle {}_NA\rangle$, therefore $a_j, a_i+a_i'-b_i\in k\langle {}_NA\rangle$ $(j\neq i)$, and $a_i'-b_i\in k\langle A_m^i\rangle\cap\varphi\langle B\rangle$, so $a_i+a_i'-b_i$ represents the same element as a_i in the range of (c), hence the homomorphism (c) is surjective. Now the proposition follows from the isomorphisms (a) and (c), and the finiteness of the set ${}_NA_m^i$. □

By Proposition 9.3 the vector space $W_0^\varphi=\sum_{i=1}^n W_i^\varphi$ together with the subspaces $W_1^\varphi,\ldots,W_n^\varphi$ define a finite dimensional n-subspace $\underline{W}^\varphi$.

PROPOSITION 9.5. *There is an additive functor* $\mathbb{S}:\mathbf{fp}(\mathbf{M}_k(\bar{T}_n))\to\mathbf{sub}_n^{\mathrm{fin}}$ *which sends* $\mathcal{M}=\mathrm{Coker}\ \varphi$ *to* $\mathbb{S}\mathcal{M}=\underline{W}^\varphi$.

Proof. By using the alternative description of controlled homomorphisms in $\mathbf{M}_k(\bar{T}_n)$ given in Section 3.1 one can easily check that the correspondence $\varphi \mapsto \underline{W}^{\varphi}$ is a functor from $\mathbf{pair}(\mathbf{M}_k(\bar{T}_n))$ to the category of n-subspaces. Furthermore, this functor factors through the natural equivalence relation $\sim$, therefore the proposition follows by Proposition 2.1. $\square$

The functors $\mathbb{S}$ and $\mathbb{M}$ are not adjoint. Moreover, one readily checks that the following proposition holds.

PROPOSITION 9.6. *The n-subspace $\mathbb{S}\mathcal{M}$ is rigid for every f. p. $\mathbf{M}_k(\bar{T}_n)$-module $\mathcal{M}$. Moreover, given a finite-dimensional n-subspace $\underline{V}$ there is a natural isomorphism $\mathbb{S}\mathbb{M}\underline{V} \simeq \underline{V}^{\mathrm{rig}}$.*

For the second statement of the proposition one uses the finite presentations constructed in the proof of Proposition 9.1.

COROLLARY 9.7. *The image of the functor $\mathbb{S}$ is the category of finite-dimensional rigid n-subspaces.*

COROLLARY 9.8. *For every f. p. $\mathbf{M}_k(\bar{T}_n)$-module $\mathcal{M}$ there is a natural isomorphism $\mathbb{S}\mathbb{M}\mathbb{S}\mathcal{M} \simeq \mathbb{S}\mathcal{M}$.*

As we pointed out in the introduction a key step to obtain a presentation of $\mathrm{Iso}(\mathbf{fp}(\mathbf{M}_k(\bar{T}_n)))$ is relating the decomposition problem in $\mathbf{fp}(\mathbf{M}_k(\bar{T}_n))$ to the decomposition problem in $\mathbf{fp}(\mathbf{M}_k(\bar{T}_1))$ and $\mathbf{sub}_n^{\mathrm{fin}}$. This is what we do in Propositions 9.9 and 9.10.

PROPOSITION 9.9. *For any f. p. $\mathbf{M}_k(\bar{T}_n)$-module $\mathcal{M}$ there exists another one $\mathcal{N}$ with $\mathbb{S}\mathcal{N}=0$ such that $\mathcal{M} \simeq \mathcal{N} \oplus \mathbb{M}\mathbb{S}\mathcal{M}$.*

Proof. Suppose that $\mathcal{M}$ is the cokernel of $\varphi : k\langle B\rangle_\beta \to k\langle A\rangle_\alpha$ in $\mathbf{M}_k(\bar{T}_n)$. By the alternative description of controlled homomorphisms given in Section 3.1 we can choose an increasing sequence of natural numbers $\{M_m\}_{m \geqslant 1}$ with $\varphi(B^i_{M_m}) \subset k\langle A^i_m\rangle$ $(1 \leqslant i \leqslant n)$. We define the inverse systems of vector spaces $X^i_\bullet, Y^i_\bullet, Z^i_\bullet$ $(1 \leqslant i \leqslant n)$ indexed by $\mathbb{N}$ in the following way

$$X^i_m = \frac{\varphi(k\langle B\rangle)}{\varphi(k\langle B^i_{M_m}\rangle)},$$

$$Y^i_m = \frac{[k\langle A^i_m\rangle + \varphi(k\langle B\rangle)] \cap \left[\sum_{j\neq i} k\langle A^j_m\rangle + \varphi(k\langle B\rangle)\right]}{\varphi(k\langle B^i_{M_m}\rangle)},$$

$$Z^i_m = \frac{[k\langle A^i_m\rangle + \varphi(k\langle B\rangle)] \cap \left[\sum_{j\neq i} k\langle A^j_m\rangle + \varphi(k\langle B\rangle)\right]}{\varphi(k\langle B\rangle)}.$$

The bonding homomorphisms are induced by the obvious inclusions of vector spaces. The short exact sequences $X^i_m \hookrightarrow Y^i_m \overset{p_i^m}{\twoheadrightarrow} Z^i_m$ are compatible with the bonding homomorphisms, so they give rise to short exact sequences $X^i_\bullet \hookrightarrow Y^i_\bullet \overset{p_i}{\twoheadrightarrow} Z^i_\bullet$ in the abelian category of pro-vector spaces. Moreover, $\lim\, Z^i_\bullet = \cap_{m \geqslant 1} Z^i_m = W^\varphi_i$. Let $\psi_i : W^\varphi_i \to Z^i_\bullet$ be the canonical pro-morphism, which is induced by the inclusions $W^\varphi_i \subset Z^i_m$. Here we regard W^φ_i as an inverse system indexed by a singleton.

The bonding homomorphisms of the inverse system $X^i_\bullet$ are surjective, therefore $\lim^1 X^i_\bullet = 0$, and by (5.b) $\mathrm{Ext}^1(W^\varphi_i, X^i_\bullet) = 0$, so there exists a pro-morphism $\tilde\psi_i$ such that the next diagram commutes

$$\begin{tikzcd}
 & & W^\varphi_i \arrow[dl, "\tilde\psi_i"'] \arrow[d, "\psi_i"] \\
X^i_\bullet \arrow[r, hook] & Y^i_\bullet \arrow[r, two heads, "p_i"] & Z^i_\bullet
\end{tikzcd}$$

The pro-morphism $\tilde\psi_i$ is represented by a sequence of homomorphisms $\tilde\psi^m_i : W^\varphi_i \to Y^i_m (m \geqslant 1)$ which are compatible with the bonding homomorphisms of $Y^i_\bullet$, and such that the composition $p^m_i \tilde\psi^m_i : W^\varphi_i \subset Z^i_m$ is the inclusion.

If $\{a^i_1, \dots, a^i_{d_i}\}$ is a basis of W^φ_i we can choose elements $\{{}_m a^i_1, \dots, {}_m a^i_{d_i}\} \subset k\langle A^i_m \rangle\, (m \geqslant 1)$ such that $\tilde\psi^m_i(a^i_j) = {}_m a^i_j + \varphi(k\langle B^i_{M_m}\rangle)$. In particular, since the homomorphisms $\tilde\psi^m_i$ are compatible with the bonding homomorphisms of $Y^i_\bullet$, we see that there are elements ${}_{m+1}b^i_j \in k\langle B^i_{M_m}\rangle$ satisfying ${}_{m+1}a^i_j - {}_m a^i_j = \varphi({}_{m+1}b^i_j)$. Moreover, let $\{a_1, \dots, a_d\}$ be a basis of $W^\varphi_0, \sigma : k\langle A\rangle/\varphi(k\langle B\rangle) \hookrightarrow k\langle A\rangle$ a splitting of the natural projection, and elements ${}_1b^i_j \in k\langle B\rangle$ such that $\varphi({}_1b^i_j) = {}_1a^i_j - \sigma({}_1a^i_j + \varphi(k\langle B\rangle))$.

If ρ is the finite presentation of $\mathbb{MS}\mathcal{M}$ constructed in the proof of Proposition 9.1, there is a morphism $\tau : \rho \to \varphi$ in $\mathbf{pair}(\mathbf{M}_k(\bar T_n))$ given by $\tau_0(w_i) = \sigma(a^i), \tau_0({}_m w^i_j) = {}_m a^i_j$, and $\tau_1({}_m w^i_j) = {}_m b^i_j$. This morphism induces a $\mathbf{M}_k(\bar T_n)$-module morphism $\tau : \mathbb{MS}\mathcal{M} \to \mathcal{M}$. Now we are going to construct a retraction of τ.

By Lemma 9.4 Z^i_m is always finite-dimensional and $W^\varphi_i = \cap_{m \geqslant 1} Z^i_m$, hence there exists $N \geqslant 1$ such that $W^\varphi_i = Z^i_N$ for every $1 \leqslant i \leqslant n$. Let $\underline{V}$ be the n-subspace given by $V_0 = k\langle A\rangle/\varphi(k\langle B\rangle)$ and $V_i = \left[k\langle A^i_N\rangle + \varphi(k\langle B\rangle)\right]/\varphi(k\langle B\rangle)$. Clearly $\mathbb{S}\mathcal{M} = \underline{V}^{\mathrm{rig}} \subset \underline{V}$, hence by Proposition 8.4 there is a retraction $r : \underline{V} \to \mathbb{S}\mathcal{M}$. By Yoneda's lemma $\mathrm{Hom}(k\langle A\rangle_\alpha, \mathbb{M}\underline{V}) = \mathbb{M}\underline{V}(k\langle A\rangle_\alpha)$. The natural projection $k\langle A\rangle \twoheadrightarrow V_0$ gives rise to a $\mathbf{M}_k(\bar T_n)$-module morphism $v_0 : k\langle A\rangle_\alpha \to \mathbb{M}\underline{V}$ such that $v_0\varphi = 0$. Moreover, since $\mathcal{M} =$Coker φ we have $\mathrm{Hom}(\mathcal{M}, \mathbb{M}\underline{V}) = \mathrm{Ker}\,\mathrm{Hom}(\varphi, \mathbb{M}\underline{V})$, in particular v_0 determines a $\mathbf{M}_k(\bar T_n)$-module morphism $v : \mathcal{M} \to \mathbb{M}\underline{V}$. One readily checks that the composite

$(\mathbb{M}r)v_0\tau_0$ coincides with the natural projection $p\colon k\langle C\rangle_\gamma \twoheadrightarrow \mathbb{M}\mathbb{S}\mathcal{M}$ = Coker ρ defined in the proof of Proposition 9.1, hence $(\mathbb{M}r)v\tau = 1$ is the identity on $\mathbb{M}\mathbb{S}\mathcal{M}$, and $(\mathbb{M}r)v$ is the desired retraction of τ. Now if we take $\mathcal{N}$ to be the cokernel of τ the proposition follows since $\mathcal{M} = \mathcal{N} \oplus \mathbb{M}\mathbb{S}\mathcal{M}$, by Corollary 9.8 $\mathbb{S}\mathcal{M} = \mathbb{S}\mathcal{N} \oplus \mathbb{S}\mathbb{M}\mathbb{S}\mathcal{M} \simeq \mathbb{S}\mathcal{N} \oplus \mathbb{S}\mathcal{M}$ and hence $\mathbb{S}\mathcal{N} = 0$. Here we use that the monoid $\mathrm{Iso}(\mathbf{sub}_n^{\mathrm{fin}})$ is free and hence cancelative, compare Section 8. □

In Proposition 9.10 we use the change of coefficients $\mathbb{F}^i_*$ associated to the additive functors $\mathbb{F}^i\colon \mathbf{M}_k(\bar{T}_1) \to \mathbf{M}_k(\bar{T}_n)$ in Remark 6.5.

PROPOSITION 9.10. *Given a f. p.* $\mathbf{M}_k(\bar{T}_n)$*-module* $\mathcal{M}$, $\mathbb{S}\mathcal{M} = 0$ *is trivial if and only if there exist f. p.* $\mathbf{M}_k(\bar{T}_1)$*-modules* $\mathcal{M}_i$ $(1 \leqslant i \leqslant n)$ *with* $\mathcal{M} \simeq \mathbb{F}^1_*\mathcal{M}_1 \oplus \cdots \oplus \mathbb{F}^n_*\mathcal{M}_n$.

Proof. It is easy to see that $\mathbb{S}\mathbb{F}^i_* = 0\,(1 \leqslant i \leqslant n)$, and $\mathbb{S}$ is additive, so the implication $\Leftarrow$ follows. Now suppose that $\mathcal{M} = \mathrm{Coker}\ [\varphi : k\langle B\rangle_\beta \to k\langle A\rangle_\alpha]$ and $\mathbb{S}\mathcal{M} = 0$. Since finite-dimensional vector spaces are artinian, by Lemma 9.4 there exists $m \geqslant 1$ big enough such that for every $1 \leqslant i \leqslant n$,

$$\frac{\left[k\langle A^i_m\rangle + \varphi(k\langle B\rangle)\right] \cap \left[\sum_{j\neq i} k\langle A^j_m\rangle + \varphi(k\langle B\rangle)\right]}{\varphi(k\langle B\rangle)} = 0,$$

that is, the following equality holds (the isomorphism on the right always holds)

$$\frac{\sum_{i=1}^n k\langle A^i_m\rangle + \varphi(k\langle B\rangle)}{\varphi(k\langle B\rangle)} = \bigoplus_{i=1}^n \frac{k\langle A^i_m\rangle + \varphi(k\langle B\rangle)}{\varphi(k\langle B\rangle)} \simeq \bigoplus_{i=1}^n \frac{k\langle A^i_m\rangle}{k\langle A^i_m\rangle \cap \varphi(k\langle B\rangle)}.$$

This is equivalent to state that

$$\left[\bigoplus_{i=1}^n k\langle A^i_m\rangle\right] \cap \varphi(k\langle B\rangle) = \bigoplus_{i=1}^n \left[k\langle A^i_m\rangle \cap \varphi(k\langle B\rangle)\right]. \tag{a}$$

By the characterization of controlled homomorphisms in Section 3.1 there exists $M \geqslant 1$ with $\varphi(B^i_M) \subset k\langle A^i_m\rangle$ $(1 \leqslant i \leqslant n)$. Let K be the kernel of the vector space homomorphism underlying to φ, that is $K = \varphi^{-1}(0)$. There is a finite set $\{b_1, \ldots, b_d\} \subset k\langle B\rangle$ which projects to a basis of

$$\frac{K + \left(\bigoplus_{i=0}^n k\langle B^i_M\rangle\right)}{\bigoplus_{i=0}^n k\langle B^i_M\rangle} \simeq \frac{K}{K \cap \left(\bigoplus_{i=0}^n k\langle B^i_M\rangle\right)},$$

since this vector space is contained in

$$\frac{k\langle B\rangle}{\bigoplus_{i=0}^n k\langle B^i_M\rangle} \simeq k\langle {}_{M-1}B\rangle,$$

and ${}_{M-1}B$ is finite.

There is also a finite set $\{a_1^i, \ldots, a_{d_i}^i\} \subset k\langle B\rangle$ which projects to a basis of

$$\frac{\varphi^{-1}(k\langle A_m^i\rangle \cap \varphi(k\langle B\rangle))}{k\langle B_M^i\rangle + K},$$

because φ induces an isomorphism

$$\frac{\varphi^{-1}(k\langle A_m^i\rangle \cap \varphi(k\langle B\rangle))}{k\langle B_M^i\rangle + K} \simeq \frac{k\langle A_m^i\rangle \cap \varphi(k\langle B\rangle)}{\varphi(k\langle B_M^i\rangle)},$$

and always

$$k\langle A_m^i\rangle \cap \left(\sum_{j\neq i} k\langle A_m^i\rangle\right) = 0, \tag{b}$$

so $k\langle A_m^i\rangle \cap \left[\sum_{j=1}^n \varphi(k\langle B_M^j\rangle)\right] = \varphi(k\langle B_M^i\rangle)$, and hence

$$\begin{aligned}\frac{k\langle A_m^i\rangle \cap \varphi(k\langle B\rangle)}{\varphi(k\langle B_M^i\rangle)} &= \frac{k\langle A_m^i\rangle \cap \varphi(k\langle B\rangle)}{k\langle A_m^i\rangle \cap \left[\sum_{j=1}^n \varphi(k\langle B_M^j\rangle)\right]} \\ &\subset \frac{\varphi(k\langle B\rangle)}{\sum_{i=1}^n \varphi(k\langle B_M^i\rangle)} \simeq \varphi(k\langle_{M-1} B\rangle).\end{aligned}$$

By (b) we have that

$$\varphi^{-1}(k\langle A_m^i\rangle \cap \varphi(k\langle B\rangle)) \cap \left[\sum_{j\neq i} \varphi^{-1}(k\langle A_m^j\rangle \cap \varphi(k\langle B\rangle))\right] = K,$$

therefore the set $\left[\sqcup_{i=1}^n \left(B_M^i \sqcup \{a_j^i\}_{j=1}^{d_i}\right)\right] \sqcup \{b_i\}_{i=1}^d$ is linearly independent in $k\langle B\rangle$; moreover it is a basis of $\sum_{i=1}^n \varphi^{-1}(k\langle A_m^i\rangle \cap \varphi(k\langle B\rangle))$. In order to complete it to a basis $\underline{B}$ of $k\langle B\rangle$ we only need to add a finite set $\{b_1', \ldots, b_{d'}'\} \subset k\langle B\rangle$ which projects to a basis of the following vector space

$$\frac{k\langle B\rangle}{\sum_{i=1}^n \varphi^{-1}(k\langle A_m^i\rangle \cap \varphi(k\langle B\rangle))}.$$

Notice that this vector space is isomorphic to

$$\frac{\varphi(k\langle B\rangle)}{\bigoplus_{i=1}^n k\langle A_m^i\rangle \cap \varphi(k\langle B\rangle)} \subset \frac{k\langle A\rangle}{\bigoplus_{i=1}^n k\langle A_m^i\rangle} \simeq k\langle_{m-1} A\rangle, \tag{c}$$

and hence finite-dimensional. The inclusion (c) follows from (a). Let $\{a_1, \ldots, a_e\} \subset k\langle A\rangle$ be a basis of

$$\frac{k\langle A\rangle}{\varphi(k\langle B\rangle) + \left(\bigoplus_{i=1}^n k\langle A_m^i\rangle\right)}.$$

By (c) $\underline{A} = (\sqcup_{i=1}^n A_m^i) \sqcup \{\varphi(b_i')\}_{i=1}^{d'} \sqcup \{a_i\}_{i=1}^{e}$ is a basis of $k\langle A\rangle$. Let $\underline{\alpha}: \underline{A} \to T_n^0, \underline{\beta}: \underline{B} \to T_n^0$ be the height functions defined as α and β over $\sqcup_{i=1}^n A_m^i$ and $\sqcup_{i=1}^n B_M^i$, respectively, and constant v_0 on the other elements. The identities $k\langle A\rangle = k\langle \underline{A}\rangle$ and $k\langle B\rangle = k\langle \underline{B}\rangle$ induce controlled isomorphisms $\phi_1: k\langle A\rangle_\alpha \simeq k\langle \underline{A}\rangle_{\underline{\alpha}}$ and $\phi_2: k\langle B\rangle_\beta \simeq k\langle \underline{B}\rangle_{\underline{\beta}}$, so if we define $\psi = \phi_1\varphi\phi_2^{-1}$ then $\mathcal{M} \simeq \operatorname{Coker}\psi$. But if we define the sets ${}^1\underline{A} = A_m^1 \sqcup \{\varphi(b_i')\}_{i=1}^{d'} \sqcup \{a_i\}_{i=1}^{e}$, ${}^i\underline{A} = A_m^i$ $(1 < i \leqslant n)$, ${}^1\underline{B} = B_M^1 \sqcup \{a_j^1\}_{j=1}^{d_1} \sqcup \{b_i\}_{i=1}^{d} \sqcup \{b_i'\}_{i=1}^{d'}$, ${}^i\underline{B} = B_M^i \sqcup \{a_j^i\}_{j=1}^{d_i}$ $(1 < i \leqslant n)$, and the height functions ${}^i\underline{\alpha}$ and ${}^i\underline{\beta}$ as the restriction of $\underline{\alpha}$ and $\underline{\beta}$ to ${}^i\underline{A}$ and ${}^i\underline{B}$, respectively $(1 \leqslant i \leqslant n)$, then we observe that $(1 \leqslant i \leqslant n)$

$$
\begin{aligned}
&{}^i\underline{\alpha}({}^i\underline{A}), {}^i\underline{\beta}({}^i\underline{B}) \subset \{v_0\} \cup \{v_m^i\}_{m \geqslant 1} \subset T_n^0,\\
&\psi({}^i\underline{B}) \subset k\langle {}^i\underline{A}\rangle,\\
&\underline{A} = \sqcup_{i=1}^n {}^i\underline{A}, \qquad \underline{B} = \sqcup_{i=1}^n {}^i\underline{B}.
\end{aligned}
$$

Notice also that $\underline{A} = \sqcup_{i=1}^n {}^i\underline{A}$ and $\underline{B} = \sqcup_{i=1}^n {}^i\underline{B}$. Hence the proposition follows. □

10. Classification of Finitely Presented $k(\boldsymbol{n})$-Modules

In this section we complete the proofs of Theorems 1.1 and 1.2 in the introduction. For this, the crucial result is Theorem 10.1 where we compute the monoid $\mathrm{Iso}(\mathbf{fp}(k(\boldsymbol{n})))$ in terms of the free abelian monoid $\mathrm{Iso}(\mathbf{sub}_n^{\mathrm{fr}})$.

THEOREM 10.1. *The following monoid morphism is an isomorphism for every $n \in \mathbb{N}$:*

$$(\Phi_n, \mathrm{Iso}(\mathbb{S})): \mathrm{Iso}(\mathbf{fp}(k(\boldsymbol{n}))) \xrightarrow{\simeq} \mathbb{N}_{\infty,n} \times \prod_{i=1}^n \mathbb{N}_\infty \times \prod_{i=1}^n \mathbb{N}_\infty \times \mathrm{Iso}(\mathbf{sub}_n^{\mathrm{fr}}).$$

This theorem follows from the strongest results previously proven in this paper. More precisely, the surjectivity of $(\Phi_n, \mathrm{Iso}(\mathbb{S}))$ is a consequence of Propositions 6.6, 7.3, Corollaries 9.2, 9.7,and 9.8 and Proposition 9.10. Furthermore, this morphism is injective by Theorem 10.2. In order to state it we introduce the following notation. Given $d \in \mathbb{N}_{\infty,n}$ the $k(\boldsymbol{n})$-module $\mathcal{A}_d$ will denote $\mathbb{F}_*^1\mathcal{A}_d$ provided $d \in \mathbb{N}_0$ and $\oplus_{i\in S}\mathbb{F}_*^i\mathcal{R}$ if $d = \infty_S$ for some $\emptyset \neq S \subset \{1, \ldots, n\}$.

THEOREM 10.2. *Any f. p. $k(\boldsymbol{n})$-module $\mathcal{M}$ decomposes in the following way*

$$\mathcal{M} \simeq \mathcal{A}_{\lambda_\mathcal{M}} \oplus \left(\bigoplus_{i=1}^n \mathbb{F}_*^i\mathcal{B}_{\mu^i_\mathcal{M}}\right) \oplus \left(\bigoplus_{i=1}^n \mathbb{F}_*^i\mathcal{C}_{\nu^i_\mathcal{M}}\right) \oplus \mathbb{MS}\mathcal{M}.$$

This theorem follows from Proposition 6.6, Theorem 7.2, Propositions 9.9 and 9.10, and the following lemma.

LEMMA 10.3. *Given two f. p. $k(\mathbf{1})$-modules $\mathcal{M}$ and $\mathcal{N}$ there exists a $k(\mathbf{n})$-module isomorphism $\mathbb{F}^i_*\mathcal{M} \simeq \mathbb{F}^j_*\mathcal{N}$ $(1 \leqslant i, j \leqslant n)$ if and only if one of the following conditions is satisfied:*

- $i = j$ *and* $\mathcal{M} \simeq \mathcal{N}$,
- $\mathcal{M} \simeq \mathcal{N} \simeq \mathcal{A}_m$ *for some* $m \in \mathbb{N}_0$.

Proof. The implication $\Rightarrow$ follows from Propositions 6.6 and 7.3 and Corollary 7.4. On the other hand if $\mathcal{M} \simeq \mathcal{N}$ then obviously $\mathbb{F}^i_*\mathcal{M} \simeq \mathbb{F}^i_*\mathcal{N}$. Moreover, $\mathbb{F}^i_*\mathcal{A}_m \simeq \mathbb{F}^j_*\mathcal{A}_m$ are isomorphic $(m \in \mathbb{N}_0)$ because both modules are isomorphic to a free $\bar{T}_n$-controlled k-vector space whose basis is a set with m elements, compare Lemma 7.7. □

As a consequence of Proposition 8.5 and Theorem 10.1 we obtain the next corollary.

COROLLARY 10.4. *The algebra $k(\mathbf{n})$ has the same representation type as the n-subspace quiver.*

Now Theorem 1.1 follows from Remarks 3.10 and 8.6 and Corollary 10.4.

We obtain from Propositions 6.6 and 7.3 and Theorems 10.1 and 10.2, the following presentation of the monoid $\mathrm{Iso}(\mathbf{fp}(k(\mathbf{n})))$.

COROLLARY 10.5. (Classification of f. p. $k(\mathbf{n})$-modules). *Let $\{\underline{V}^{(n,j)}\}_{j \in J_n}$ be the set of indecomposable rigid n-subspaces. There is a solution to the decomposition problem in the category of f. p. $k(\mathbf{n})$-modules given by the following $1 + 5n +$ card J_n elementary modules* $(1 \leqslant i \leqslant n,\ j \in J_n)$

$$\mathbb{F}^1_*\mathcal{A}, \mathbb{F}^i_*\mathcal{R}, \mathbb{F}^i_*\mathcal{B}, \mathbb{F}^i_*\mathcal{B}_\infty, \mathbb{F}^i_*\mathcal{C}, \mathbb{F}^i_*\mathcal{C}_\infty, \mathbb{M}\underline{V}^{(n,j)},$$

and $6n$ *elementary isomorphisms* $(1 \leqslant i \leqslant n)$

$$\begin{aligned} &\mathbb{F}^1_*\mathcal{A} \oplus \mathbb{F}^i_*\mathcal{R} \simeq \mathbb{F}^i_*\mathcal{R}, \qquad \mathbb{F}^i_*\mathcal{R} \oplus \mathbb{F}^i_*\mathcal{R} \simeq \mathbb{F}^i_*\mathcal{R}, \qquad \mathbb{F}^i_*\mathcal{B} \oplus \mathbb{F}^i_*\mathcal{B}_\infty \simeq \mathbb{F}^i_*\mathcal{B}_\infty, \\ &\mathbb{F}^i_*\mathcal{B}_\infty \oplus \mathbb{F}^i_*\mathcal{B}_\infty \simeq \mathbb{F}^i_*\mathcal{B}_\infty, \qquad \mathbb{F}^i_*\mathcal{C} \oplus \mathbb{F}^i_*\mathcal{C}_\infty \simeq \mathbb{F}^i_*\mathcal{C}_\infty, \qquad \mathbb{F}^i_*\mathcal{C}_\infty \oplus \mathbb{F}^i_*\mathcal{C}_\infty \simeq \mathbb{F}^i_*\mathcal{C}_\infty. \end{aligned}$$

This classification theorem together with Proposition 8.7 and Remark 8.8 complete the proof of Theorem 1.2.

Appendix A. Some Computations of $\mathrm{Ext}^1_{k(\boldsymbol{n})}$ Groups

The aim of this appendix is to provide some tools and techniques to compute the $\mathrm{Ext}^1_{k(\boldsymbol{n})}$ group of any pair of f. p. $k(\boldsymbol{n})$-modules. This group is in fact a k-vector space, so it is determined by its dimension. Higher $\mathrm{Ext}^*_{k(\boldsymbol{n})}$ groups vanish over f. p. $k(\boldsymbol{n})$-modules by Corollary 7.16. Since the functor $\mathrm{Ext}^1_{k(\boldsymbol{n})}$ is biadditive we just have to compute it over pairs of elementary f. p. $k(\boldsymbol{n})$-modules (see Corollay 10.5). We shall not make all these computations here for an arbitrary n, but just for $n=1$, $k(\boldsymbol{1})=\mathrm{RCFM}(k)$. In addition we show for any $n\in\mathbb{N}$ that the $\mathrm{Ext}^1_{k(\boldsymbol{n})}$ of pairs of f. p. $k(\boldsymbol{n})$-modules coming from finite-dimensional n-subspaces via the functor $\mathbb{M}$ in (9.a) coincide with their $\mathrm{Ext}^1_{kQ_n}$ as modules over the path algebra. This last vector space is much easier to compute, since one can use the integral bilinear form of the quiver Q_n (see [17]).

Let $\mathcal{R}$ be the k-algebra RCFM(k) as in Section 7. Given two elementary $\mathcal{R}$-modules $\mathcal{R}/\mathsf{Y}\mathcal{R}\,(\mathsf{Y}\neq 0)$ and $\mathcal{R}/\mathsf{Z}\mathcal{R}$ (see Theorem 7.1), one can check by using Lemma 7.9 and basic homological algebra that there is an isomorphism of k-vector spaces

$$\mathrm{Ext}^1_{\mathcal{R}}(\mathcal{R}/\mathsf{Y}\mathcal{R},\mathcal{R}/\mathsf{Z}\mathcal{R})\simeq\frac{\mathcal{R}}{\mathcal{R}\mathsf{Y}+\mathsf{Z}\mathcal{R}}. \tag{A.a}$$

This formula also holds for $\mathsf{Z}=0$, moreover, in this case it is a left-$\mathcal{R}$ -module isomorphism.

LEMMA A.1. *We have the following identities*

(1) $\mathsf{A}\mathcal{R}=\{\mathsf{R}\in\mathcal{R};\ \mathsf{r}_{0j}=0$ *for all* $j\in\mathbb{N}_0\}$,
(2) $(\mathsf{I}-\mathsf{A})\mathcal{R}=\{\mathsf{R}\in\mathcal{R};\ \sum_{i\in\mathbb{N}_o}\mathsf{r}_{ij}=0$ *for all* $j\in\mathbb{N}_0\}$,
(3) $(\mathsf{I}-\mathsf{A}^t)\mathcal{R}=\{\mathsf{R}\in\mathcal{R};$ *given any* $i\in\mathbb{N}_0$, $\sum_{n\geqslant i}\mathsf{r}_{nj}=0$ *for almost all* $j\in\mathbb{N}_0\}$,
(4) $(\mathsf{I}-\mathsf{B}^t)\mathcal{R}=\{\mathsf{R}\in\mathcal{R};$ *given* $m\in\mathbb{N}_0$ *and* $i\leqslant m$, $\sum_{n\geqslant m}\mathsf{r}_{i+\frac{n(n+1)}{2},j}=0$ *for almost all* $j\in\mathbb{N}_0\}$.

Proof. One can check that the right-hand-side sets of the statement are ideals, hence in order to establish the inclusions $\subset$ it is enough to prove that the matrix defining each left-hand-side ideal belongs to the corresponding right-hand-side set. This can be checked by a tedious but straightforward computation. Suppose now that R is a matrix in the right-hand-side set of (1), (2), (3), or (4), then one can check that the matrix $\mathsf{C}^1,\mathsf{C}^2,\mathsf{C}^3$ or C^4 defined as $\mathsf{c}^1_{0j}=0$, $\mathsf{c}^1_{i+1,j}=\mathsf{r}_{ij}$, $\mathsf{c}^2_{ij}=\sum_{n=0}^{i}\mathsf{r}_{nj}$, $\mathsf{c}^3_{ij}=\sum_{n\geqslant i}\mathsf{r}_{nj}$ $(i,j\in\mathbb{N}_0)$, $\mathsf{c}^4_{i+\frac{m(m+1)}{2},j}=\sum_{n\geqslant m}\mathsf{r}_{i+\frac{n(n+1)}{2},j}$ $(i,j\in\mathbb{N}_0, m\geqslant i)$, belongs to $\mathcal{R}$ and satisfies $\mathsf{A}\mathsf{C}^1=\mathsf{R}$, $(\mathsf{I}-\mathsf{A})\mathsf{C}^2=\mathsf{R}$, $(\mathsf{I}-\mathsf{A}^t)\mathsf{C}^3=\mathsf{R}$ or $(\mathsf{I}-\mathsf{B}^t)\mathsf{C}^4=\mathsf{R}$, provided we are in case (1), (2), (3) or (4). Hence we are done. $\square$

By using Lemma A.1 and the involution of the k-vector space $\mathcal{R}$ given by transposition of matrices, one readily checks that the following lemma.

LEMMA A.2. *We have the following equalities:*

(1) $\mathcal{R}(\mathsf{I}-\mathsf{A})=\left\{\mathsf{R}\in\mathcal{R};\ \textit{given any } j\in\mathbb{N}_0, \sum_{n\geqslant j}\mathsf{r}_{in}=0 \textit{ for almost all } i\in\mathbb{N}_0\right\}$,

(2) $\mathcal{R}(\mathsf{I}-\mathsf{A}^t) = \left\{\mathsf{R}\in\mathcal{R}; \sum_{j\in\mathbb{N}_0}\mathsf{r}_{ij}=0 \textit{ for all } i\in\mathbb{N}_0\right\}$,

(3) $\mathcal{R}(\mathsf{I}-\mathsf{B})=\left\{\mathsf{R}\in\mathcal{R};\ \textit{given } m\in\mathbb{N}_0 \textit{ and } j\leqslant m, \sum_{n\geqslant m}\mathsf{r}_{i,j+\frac{n(n+1)}{2}}=0 \textit{ for almost all } i\in\mathbb{N}_0\right\}$.

PROPOSITION A.3. *We have that*

(1) $\dim \operatorname{Ext}^1_{\mathcal{R}}(\mathcal{B},\mathcal{R})=(\text{card } k)^{\aleph_0}$,
(2) $\dim \operatorname{Ext}^1_{\mathcal{R}}(\mathcal{C},\mathcal{R})=(\text{card } k)^{\aleph_0}$,
(3) $\dim \operatorname{Ext}^1_{\mathcal{R}}(\mathcal{B}_\infty,\mathcal{R})=(\text{card } k)^{\aleph_0}$,
(4) $\dim \operatorname{Ext}^1_{\mathcal{R}}(\mathcal{C}_\infty,\mathcal{R})=(\text{card } k)^{\aleph_0}$,
(5) $\dim \operatorname{Ext}^1_{\mathcal{R}}(\mathcal{B},\mathcal{A})=0$,
(6) $\dim \operatorname{Ext}^1_{\mathcal{R}}(\mathcal{C},\mathcal{A})=1$,
(7) $\dim \operatorname{Ext}^1_{\mathcal{R}}(\mathcal{B}_\infty,\mathcal{A})=0$,
(8) $\dim \operatorname{Ext}^1_{\mathcal{R}}(\mathcal{C}_\infty,\mathcal{A})=\aleph_0$.

Proof. One can check by using (A.2) (1), (A.1) (1) and (A.2) (2), and (A.1) (1) and (A.2) (3), that there are k-vector space isomorphisms

$$\frac{\mathcal{R}}{\mathcal{R}(\mathsf{I}-\mathsf{A})}\simeq\frac{\prod_{i\in\mathbb{N}_0}k}{\bigoplus_{i\in\mathbb{N}_0}k},\qquad \mathsf{R}+\mathcal{R}(\mathsf{I}-\mathsf{A})\mapsto\left(\sum_{j\in\mathbb{N}_0}\mathsf{r}_{ij}\right)_{i\in\mathbb{N}_0}+\bigoplus_{i\in\mathbb{N}_0}k,$$

$$\frac{\mathcal{R}}{\mathcal{R}(\mathsf{I}-\mathsf{A}^t)+\mathsf{A}\mathcal{R}}\simeq k,\qquad \mathsf{R}+(\mathcal{R}(\mathsf{I}-\mathsf{A}^t)+\mathsf{A}\mathcal{R})\mapsto\sum_{j\in\mathbb{N}_0}\mathsf{r}_{0j},$$

$$\frac{\mathcal{R}}{\mathcal{R}(\mathsf{I}-\mathsf{B}^t)+\mathsf{A}\mathcal{R}}\simeq\bigoplus_{j\in\mathbb{N}_0}k,\qquad \mathsf{R}+(\mathcal{R}(\mathsf{I}-\mathsf{B}^t)+\mathsf{A}\mathcal{R})\mapsto\left(\sum_{n\geqslant j}\mathsf{r}_{0,j+\frac{n(n+1)}{2}}\right)_{j\in\mathbb{N}_0},$$

hence (1), (6) and (8) follow from (A.a).

For any pair of elementary f. p. $\mathcal{R}$-modules the inequality $\dim\operatorname{Ext}^1_{\mathcal{R}}\leqslant(\text{card } k)^{\aleph_0}$ follows from (A.a) and Proposition 3.8. Now (2) is a consequence of (1) and Proposition 7.10 (1). Moreover (3) follows from (1) and the fact that $\mathcal{B}$ is a direct summand of $\mathcal{B}_\infty$, see Theorem 7.1, and (4) is a consequence of (3) and Proposition 7.10 (2).

Given any matrix $\mathsf{R}\in\mathcal{R}$, if $\mathsf{R}^1,\mathsf{R}^2\in\mathcal{R}$ are the matrices defined by $\mathsf{r}^1_{0j}=\mathsf{r}_{0j}, \mathsf{r}^2_{ij}=\mathsf{r}_{ij}$ $(i>0)$, and $\mathsf{r}^n_{ij}=0$ otherwise, then $\mathsf{R}=\mathsf{R}^1+\mathsf{R}^2, \mathsf{R}^1\in\mathcal{R}(\mathsf{I}-\mathsf{B})$ and $\mathsf{R}^2\in\mathsf{A}\mathcal{R}$, by (A.1) (1) and (A.2) (3), hence $\mathcal{R}=\mathcal{R}(\mathsf{I}-\mathsf{B})+\mathsf{A}\mathcal{R}$ and (7) follows by (A.a). Moreover, since $\mathcal{B}$ is a direct summand of $\mathcal{B}_\infty$ by Theorem 7.1 then (5) also follows. □

By Theorem 7.1, Corollaries 7.12 and 7.17 the $\mathrm{Ext}^1_{\mathcal{R}}$ group of any other pair of elementary f. p. $\mathcal{R}$-modules is zero, hence the first extension groups are now completely determined for f. p. $\mathcal{R}$-modules.

PROPOSITION A.4. *For any pair of finite-dimensional n-subspaces there is a natural isomorphism*

$$\mathrm{Ext}^1_{kQ_n}(\underline{V}, \underline{W}) \simeq \mathrm{Ext}^1_{k(\boldsymbol{n})}(\mathbb{M}\underline{V}, \mathbb{M}\underline{W}).$$

proof. Projective representations of Q_n are (arbitrary) direct sums of the following $n+1$ indecomposable n-subspaces,

$$\mathbb{F}^1(0 \to k), \qquad \mathbb{F}^i(k \to k), \qquad (1 \leqslant i \leqslant n).$$

Since $\mathbb{M}$ is an exact full inclusion of categories and any finite-dimensional representation of Q_n admits a length-one projective resolution by finite-dimensional projective representations it is enough to check that

(1) $\mathrm{Ext}^1_{k(\boldsymbol{n})}(\mathbb{M}\mathbb{F}^1(0 \to k), \mathbb{M}\mathbb{F}^1(0 \to k)) = 0$,
(2) $\mathrm{Ext}^1_{k(\boldsymbol{n})}(\mathbb{M}\mathbb{F}^1(0 \to k), \mathbb{M}\mathbb{F}^i(k \to k)) = 0$ $(1 \leqslant i \leqslant n)$,
(3) $\mathrm{Ext}^1_{k(\boldsymbol{n})}(\mathbb{M}\mathbb{F}^i(k \to k), \mathbb{M}\mathbb{F}^1(0 \to k)) = 0$ $(1 \leqslant i \leqslant n)$,
(4) $\mathrm{Ext}^1_{k(\boldsymbol{n})}(\mathbb{M}\mathbb{F}^i(k \to k), \mathbb{M}\mathbb{F}^j(k \to k)) = 0$ $(1 \leqslant i, j \leqslant n)$.

The resolution constructed in the proof of Proposition 9.1 shows that $\mathbb{M}\mathbb{F}^1(0 \to k)$ is a projective $k(\boldsymbol{n})$-module isomorphic to a 1-dimensional $\bar{T}_n$-controlled k-vector space, hence (1) and (2) hold. Moreover, one can easily check (3) and (4) by using the definition of $\mathbb{M}$ in (9.a) and the resolutions of $\mathbb{M}\mathbb{F}^i(k \to k) = 0$ $(1 \leqslant i \leqslant n)$ in the proof of Proposition 9.1. □

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